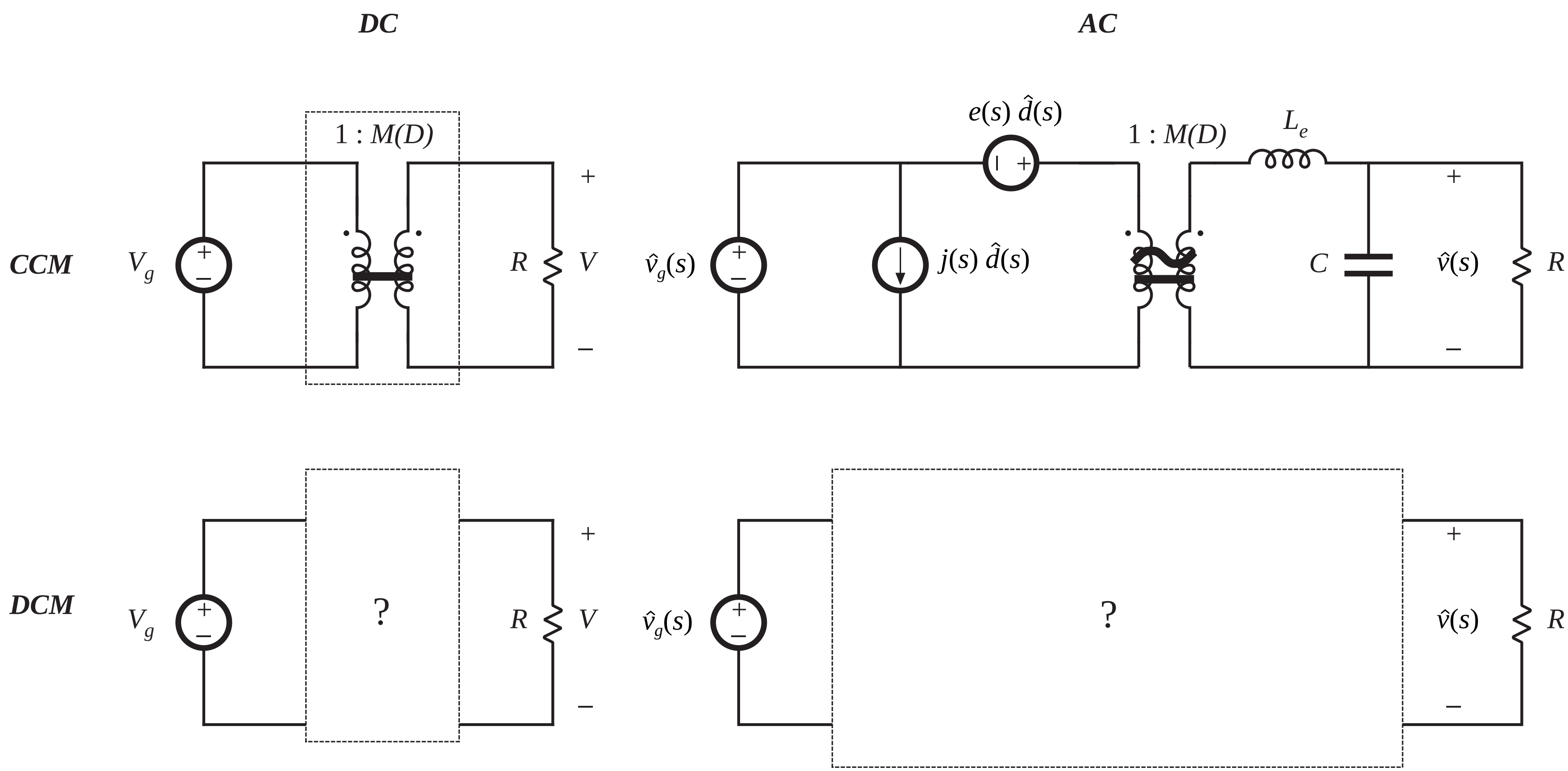
Chapter 10
Ac and Dc Equivalent Circuit Modeling of the Discontinuous Conduction Mode

We are missing ac and dc equivalent circuit models for the discontinuous conduction mode



Change in characteristics at the CCM / DCM boundary

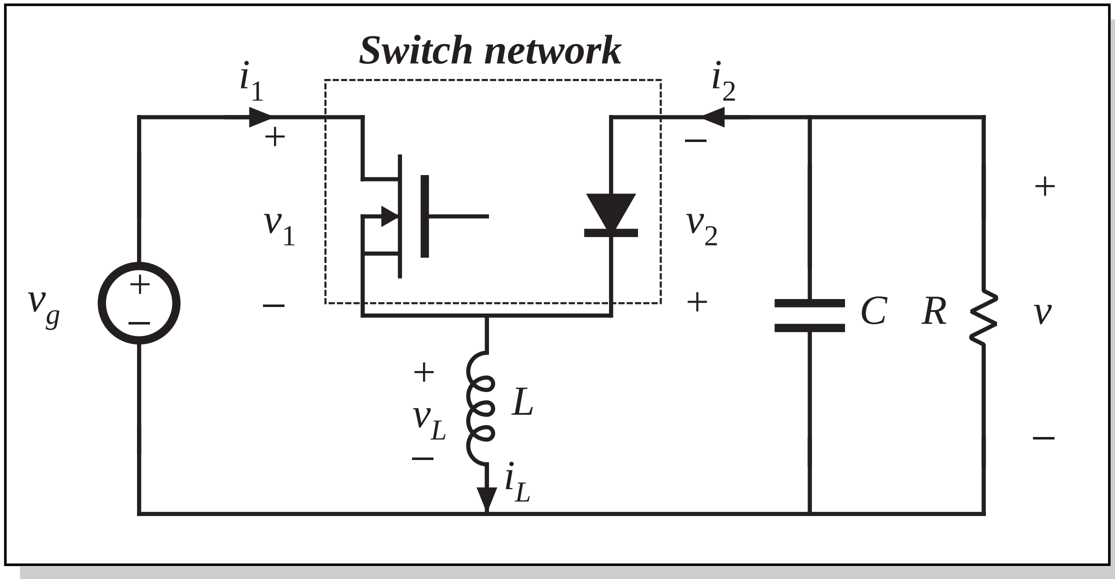
- Steady-state output voltage becomes strongly load-dependent
- Simpler dynamics: one pole and the RHP zero are moved to very high frequency, and can normally be ignored
- Traditionally, boost and buck-boost converters are designed to operate in DCM at full load
- All converters may operate in DCM at light load

So we need equivalent circuits that model the steady-state and small-signal ac models of converters operating in DCM

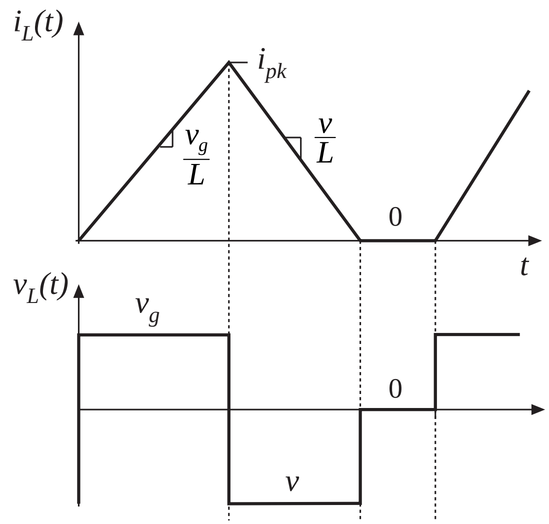
The averaged switch approach will be employed

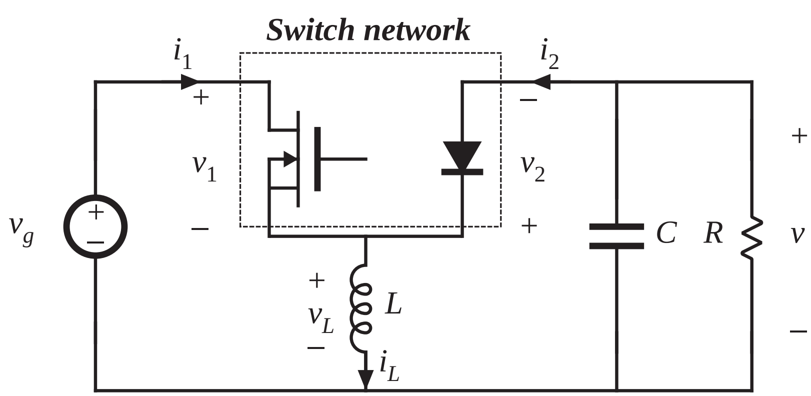
10.1 Derivation of DCM averaged switch model: buck-boost example

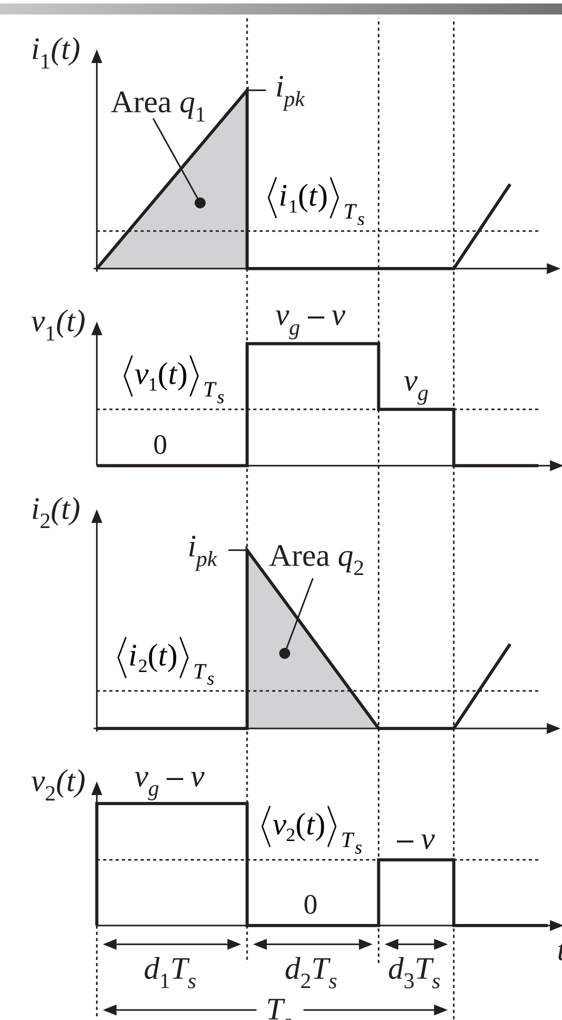
- Define switch terminal quantities v_1, i_1, v_2, i_2, as shown
- Let us find the averaged quantities $\langle v_1 \rangle$, $\langle i_1 \rangle$, $\langle v_2 \rangle$, $\langle i_2 \rangle$, for operation in DCM, and determine the relations between them



DCM waveforms







Basic DCM equations

Peak inductor current:

$$i_{pk} = \frac{v_g}{L} d_1 T_s$$

Average inductor voltage:

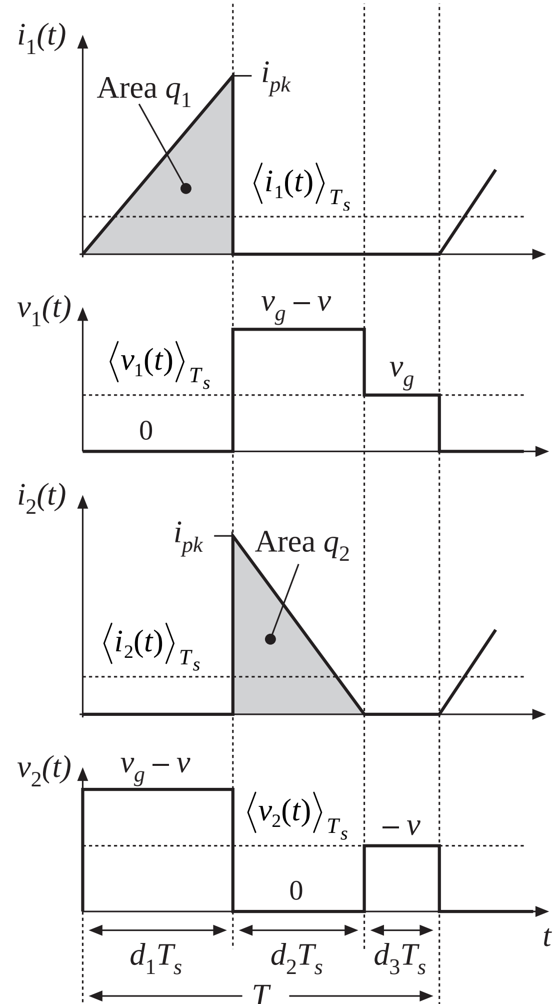
$$\left\langle v_L(t) \right\rangle_{T_s} = d_1 \left\langle v_g(t) \right\rangle_{T_s} + d_2 \left\langle v(t) \right\rangle_{T_s} + d_3 \cdot 0$$

In DCM, the diode switches off when the inductor current reaches zero. Hence, $i(0) = i(T_s) = 0$, and the average inductor voltage is zero. This is true even during transients.

$$\left\langle v_L(t) \right\rangle_{T_s} = d_1(t) \left\langle v_g(t) \right\rangle_{T_s} + d_2(t) \left\langle v(t) \right\rangle_{T_s} = 0$$

Solve for d_2:

$$d_2(t) = -\, d_1(t) \frac{\left\langle v_g(t) \right\rangle_{T_s}}{\left\langle v(t) \right\rangle_{T_s}}$$



Average switch network terminal voltages

Average the $v_1(t)$ waveform:

$$\left\langle v_1(t) \right\rangle_{T_s} = d_1(t) \cdot 0 + d_2(t) \left(\left\langle v_g(t) \right\rangle_{T_s} - \left\langle v(t) \right\rangle_{T_s} \right) + d_3(t) \left\langle v_g(t) \right\rangle_{T_s}$$

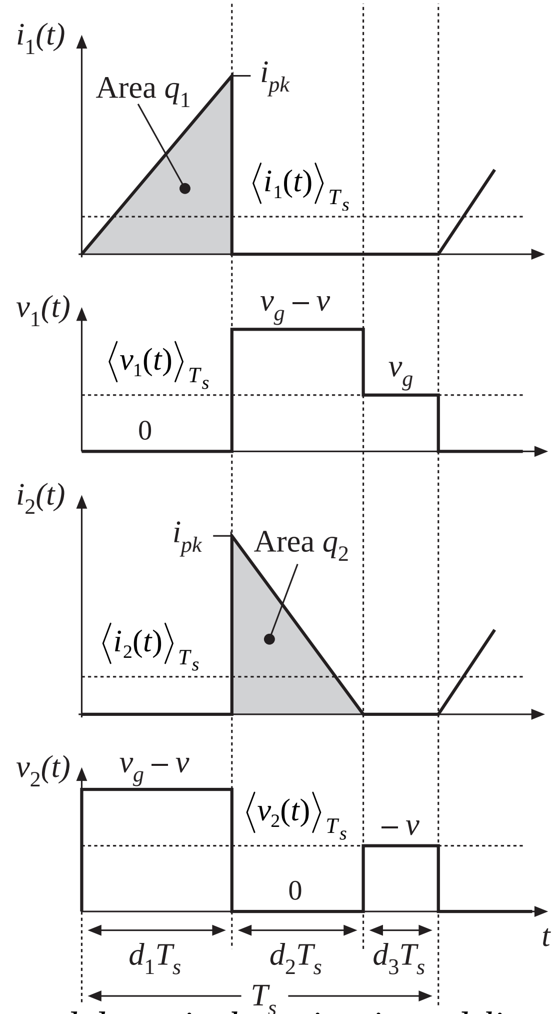
Eliminate d_2 and d_3:

$$\left\langle v_1(t) \right\rangle_{T_s} = \left\langle v_g(t) \right\rangle_{T_s}$$

Similar analysis for $v_2(t)$ waveform leads to

$$\left\langle v_2(t) \right\rangle_{T_s} = d_1(t) \left(\left\langle v_g(t) \right\rangle_{T_s} - \left\langle v(t) \right\rangle_{T_s} \right) + d_2(t) \cdot 0 + d_3(t) \left(- \left\langle v(t) \right\rangle_{T_s} \right)$$

$$= - \left\langle v(t) \right\rangle_{T_s}$$



Average switch network terminal currents

Average the $i_1(t)$ waveform:

$$\left\langle i_1(t) \right\rangle_{T_s} = \frac{1}{T_s} \int_t^{t+T_s} i_1(t) dt = \frac{q_1}{T_s}$$

The integral q_1 is the area under the $i_1(t)$ waveform during first subinterval. Use triangle area formula:

$$q_1 = \int_t^{t+T_s} i_1(t) dt = \frac{1}{2} \left(d_1 T_s \right) \left(i_{pk} \right)$$

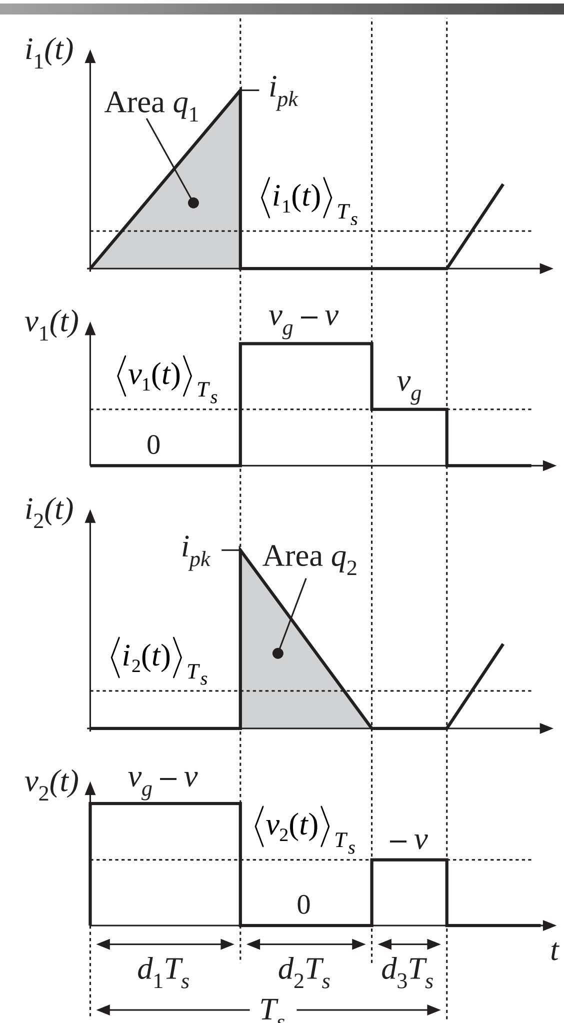
Eliminate i_{pk}:

$$\left\langle i_1(t) \right\rangle_{T_s} = \frac{d_1^2(t)\ T_s}{2L} \left\langle v_1(t) \right\rangle_{T_s}$$

Note $\langle i_1(t) \rangle_{Ts}$ is not equal to $d\ \langle i_L(t) \rangle_{Ts}$!

Similar analysis for $i_2(t)$ waveform leads to

$$\left\langle i_2(t) \right\rangle_{T_s} = \frac{d_1^2(t)\ T_s}{2L} \frac{\left\langle v_1(t) \right\rangle_{T_s}^2}{\left\langle v_2(t) \right\rangle_{T_s}}$$

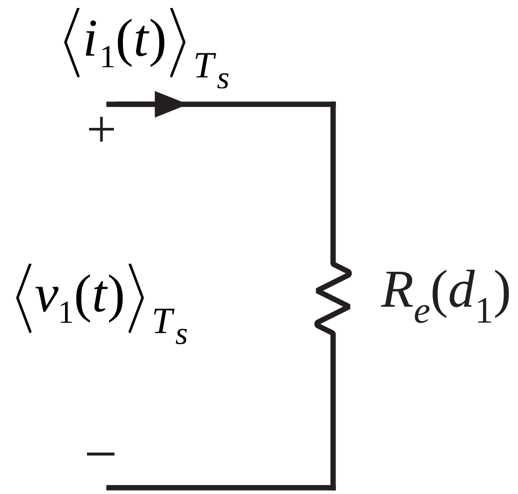


Input port: Averaged equivalent circuit

$$\left\langle i_1(t) \right\rangle_{T_s} = \frac{d_1^2(t)\, T_s}{2L} \left\langle v_1(t) \right\rangle_{T_s}$$

$$\left\langle i_1(t) \right\rangle_{T_s} = \frac{\left\langle v_1(t) \right\rangle_{T_s}}{R_e(d_1)}$$

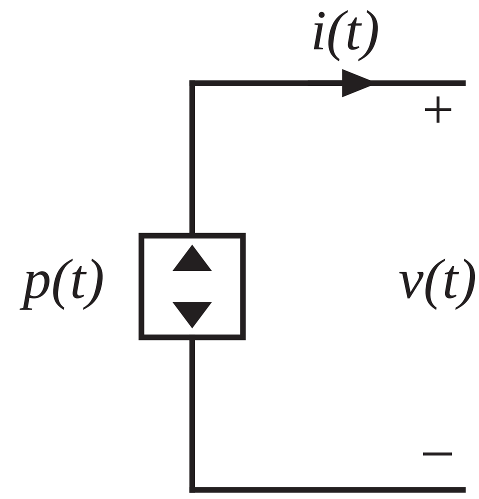
$$R_e(d_1) = \frac{2L}{d_1^2\, T_s}$$



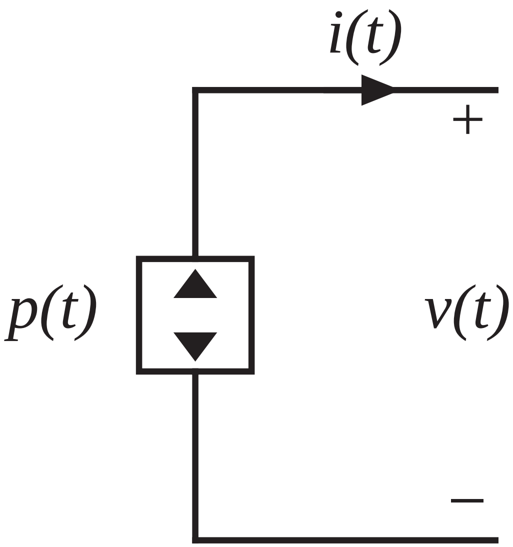
Output port: Averaged equivalent circuit

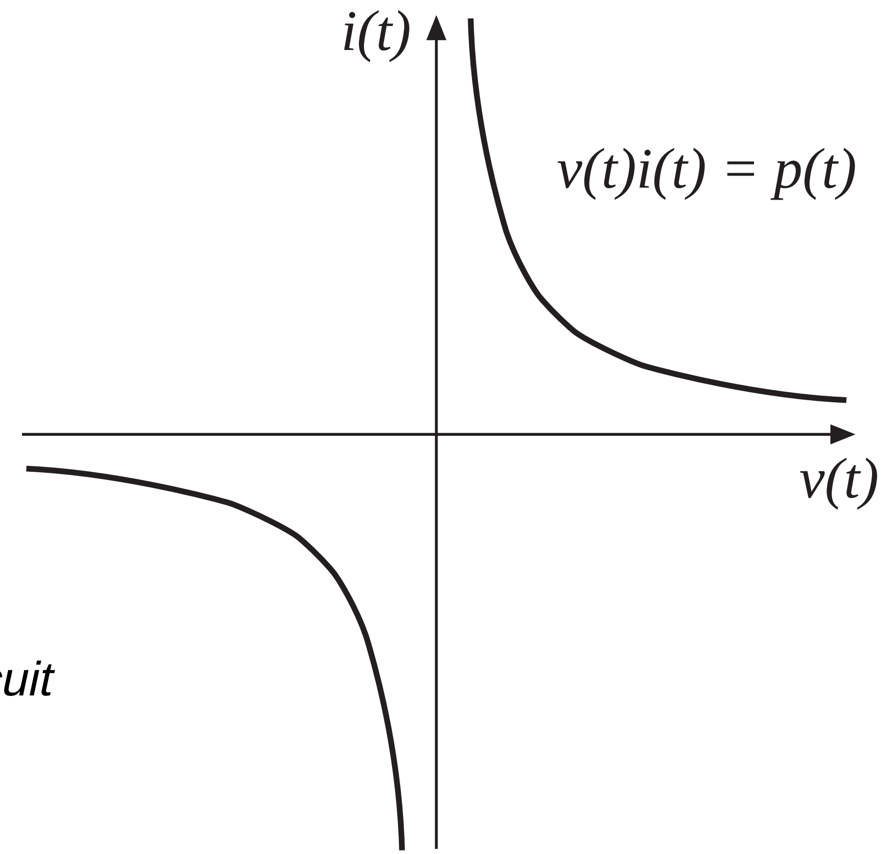
$$\left\langle i_2(t) \right\rangle_{T_s} = \frac{d_1^2(t)\, T_s}{2L} \frac{\left\langle v_1(t) \right\rangle_{T_s}^2}{\left\langle v_2(t) \right\rangle_{T_s}}$$

$$\left\langle i_2(t) \right\rangle_{T_s} \left\langle v_2(t) \right\rangle_{T_s} = \frac{\left\langle v_1(t) \right\rangle_{T_s}^2}{R_e(d_1)} = \left\langle p(t) \right\rangle_{T_s}$$



The dependent power source





- *Must avoid open- and short-circuit connections of power sources*
- *Power sink: negative* $p(t)$

How the power source arises in lossless two-port networks

In a lossless two-port network without internal energy storage: instantaneous input power is equal to instantaneous output power

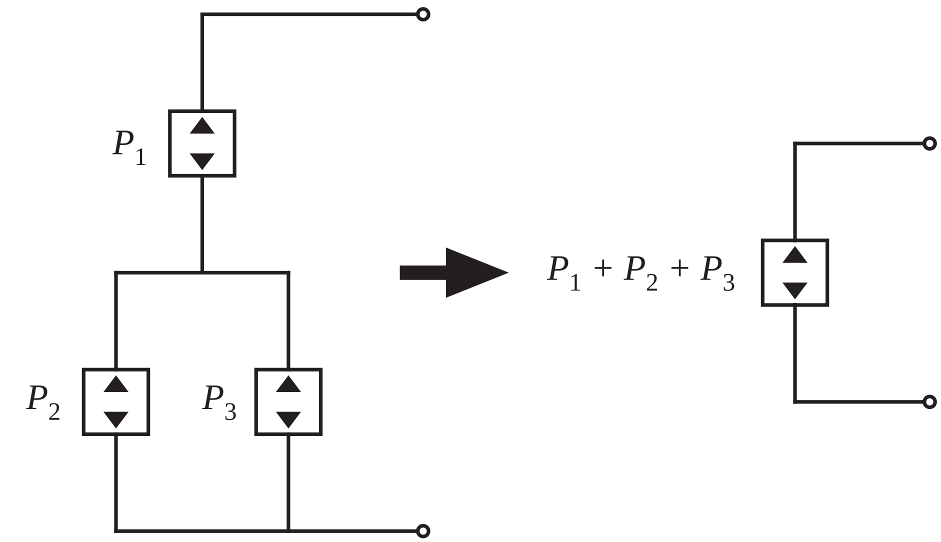
In all but a small number of special cases, the instantaneous power throughput is dependent on the applied external source and load

If the instantaneous power depends only on the external elements connected to one port, then the power is not dependent on the characteristics of the elements connected to the other port. The other port becomes a source of power, equal to the power flowing through the first port

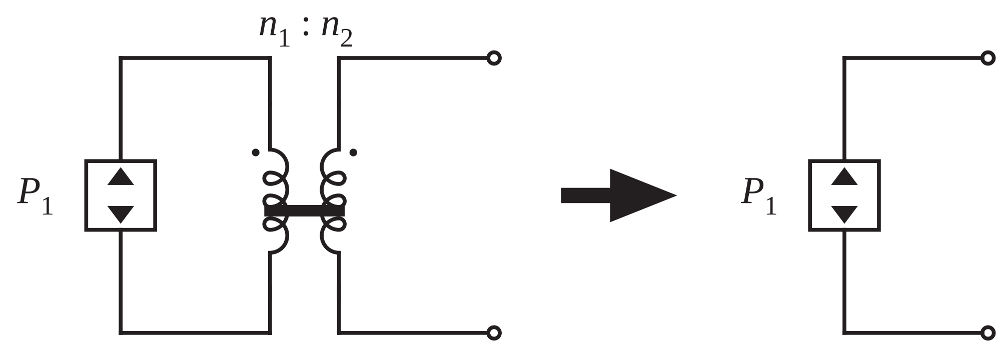
A power source (or power sink) element is obtained

Properties of power sources

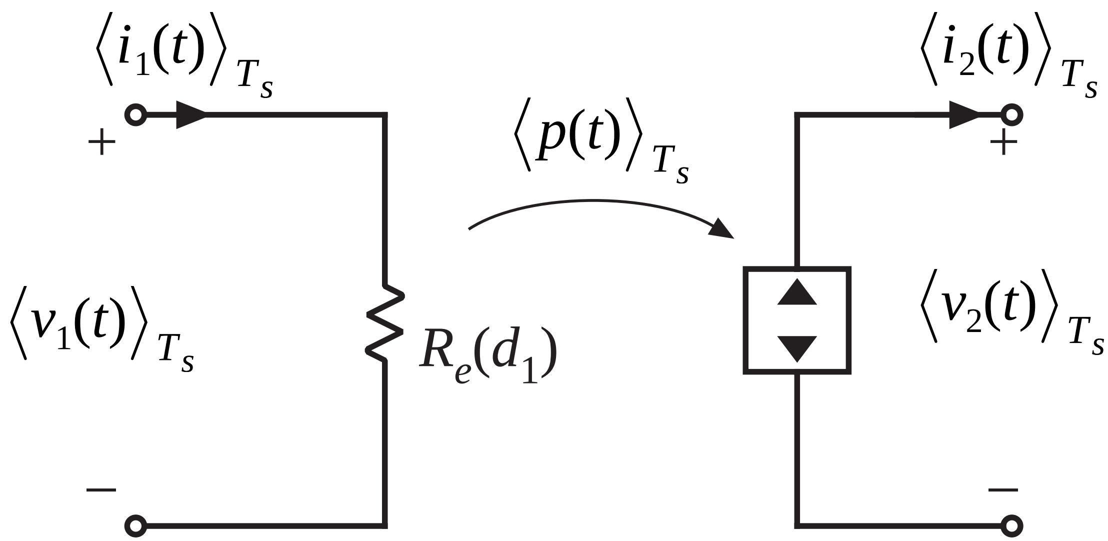
Series and parallel connection of power sources



Reflection of power source through a transformer



The loss-free resistor (LFR)



A two-port lossless network

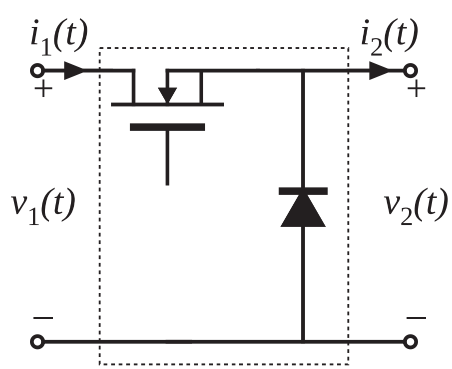
Input port obeys Ohm's Law

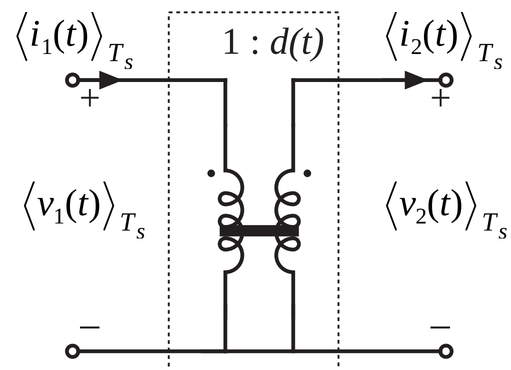
Power entering input port is transferred to output port

Averaged modeling of CCM and DCM switch networks

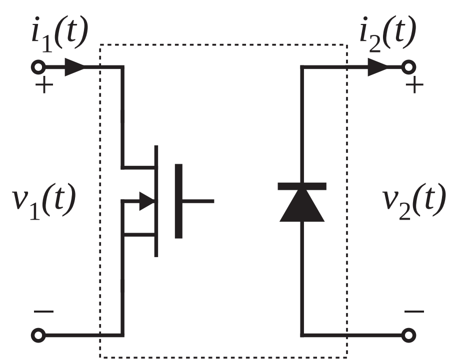
Switch network ***Averaged switch model***

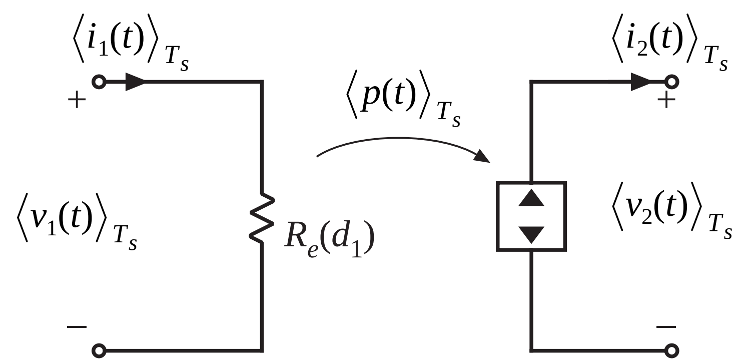
CCM





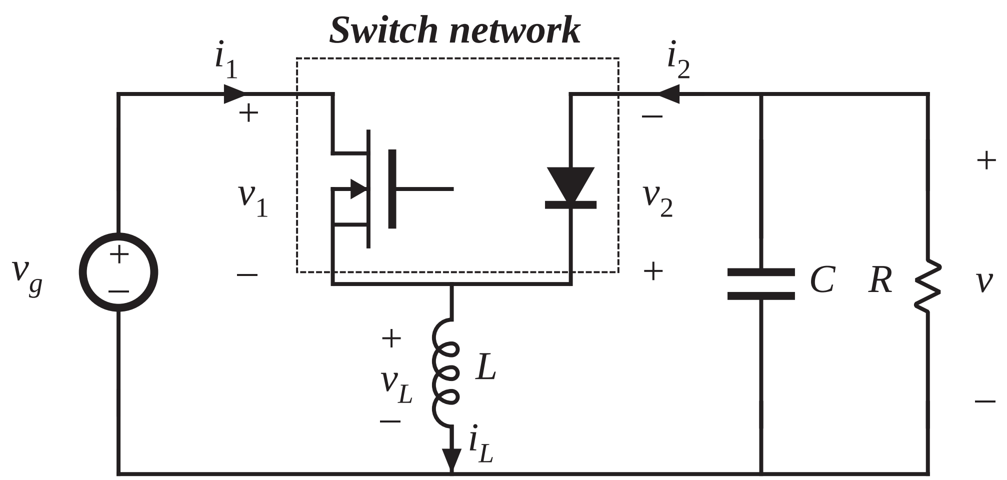
DCM



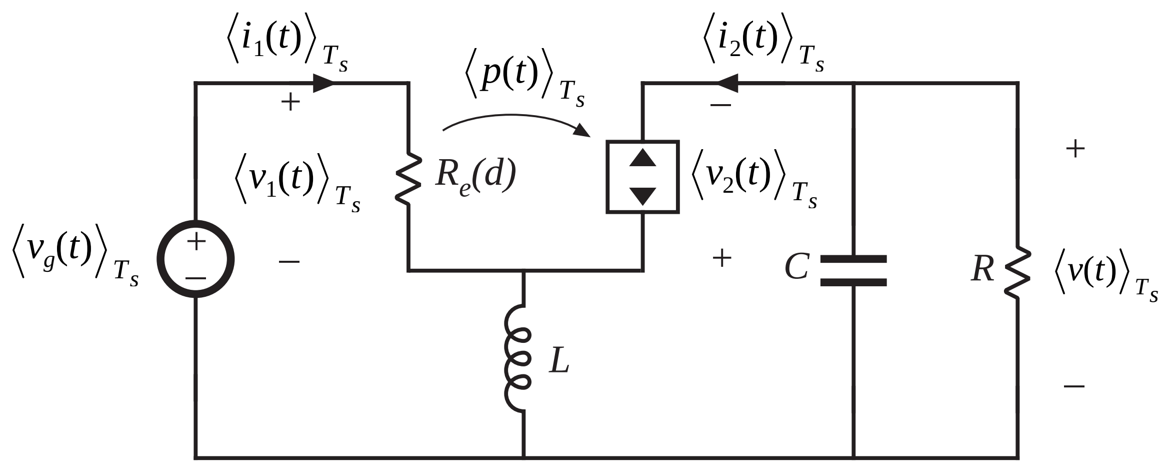


Averaged switch model: buck-boost example

Original circuit



Averaged model

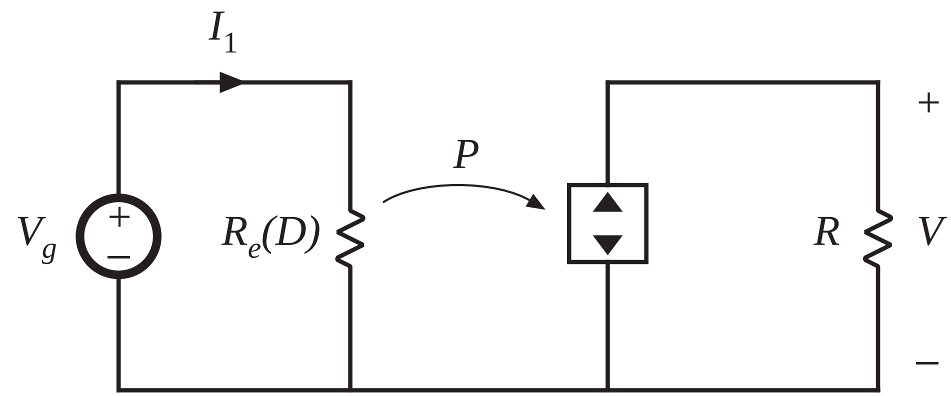


Solution of averaged model: steady state

Let

$L \rightarrow$ short circuit

$C \rightarrow$ open circuit



Converter input power:

$$P = \frac{V_g^2}{R_e}$$

Converter output power:

$$P = \frac{V^2}{R}$$

Equate and solve:

$$P = \frac{V_g^2}{R_e} = \frac{V^2}{R}$$

$$\frac{V}{V_g} = \pm\sqrt{\frac{R}{R_e}}$$

Steady-state LFR solution

$$\frac{V}{V_g} = \pm\sqrt{\frac{R}{R_e}}$$

is a general result, for any system that can be modeled as an LFR.

For the buck-boost converter, we have

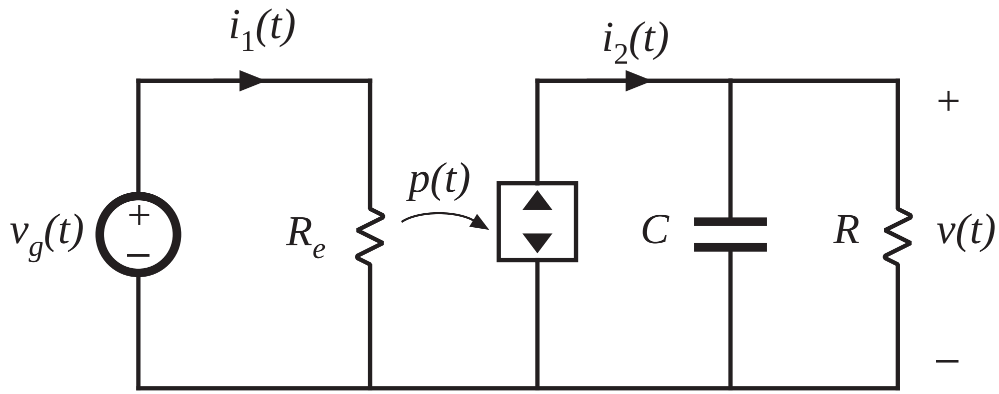
$$R_e(D) = \frac{2L}{D^2 T_s}$$

Eliminate R_e:

$$\frac{V}{V_g} = -\sqrt{\frac{D^2 T_s R}{2L}} = -\frac{D}{\sqrt{K}}$$

which agrees with the previous steady-state solution of Chapter 5.

Steady-state LFR solution with ac terminal waveforms



Converter average input power:

$$P_{av} = \frac{V_{g,rms}^2}{R_e}$$

Converter average output power:

$$P_{av} = \frac{V_{rms}^2}{R}$$

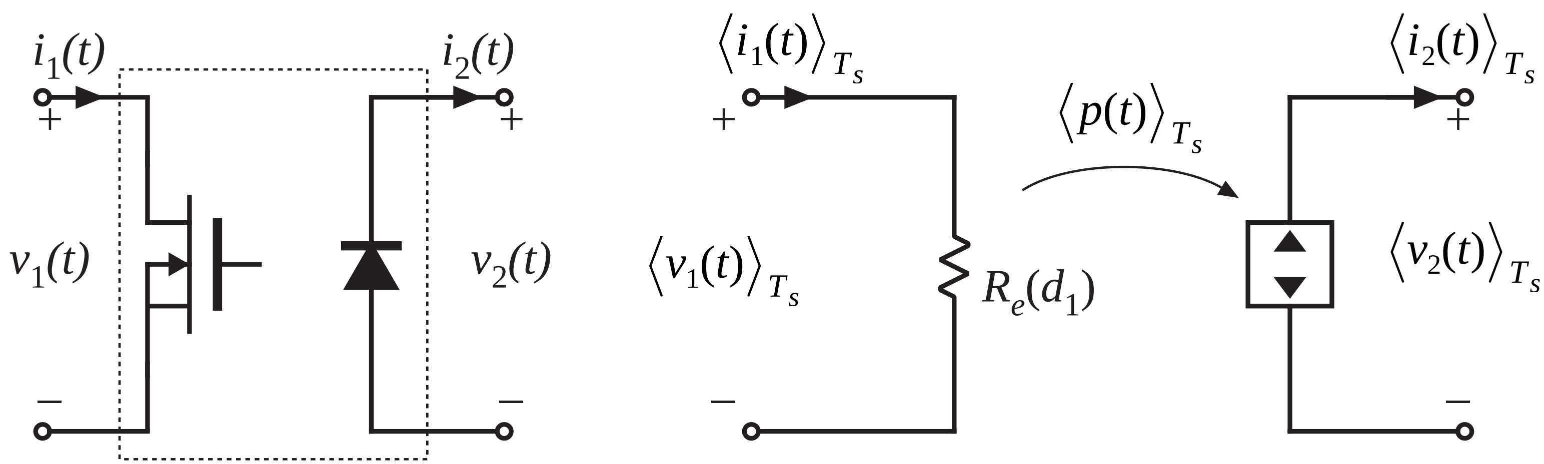
Note that no average power flows into capacitor

Equate and solve:

$$\frac{V_{rms}}{V_{g,rms}} = \sqrt{\frac{R}{R_e}}$$

Averaged models of other DCM converters

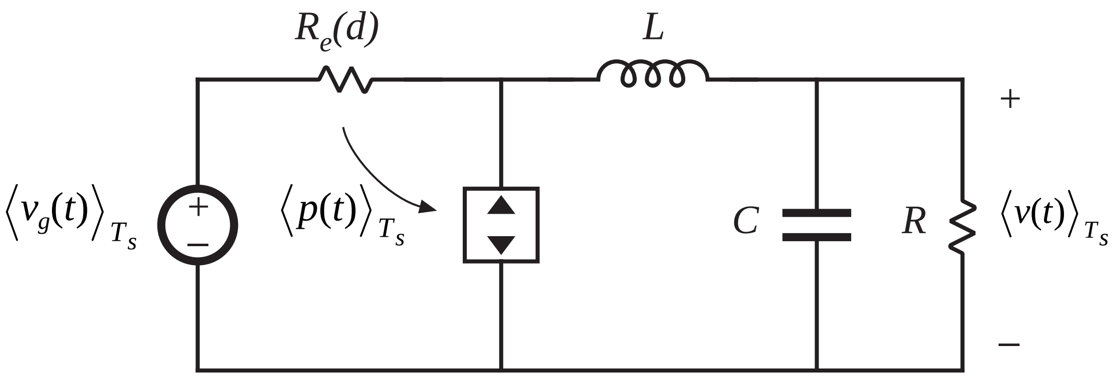
- Determine averaged terminal waveforms of switch network
- In each case, averaged transistor waveforms obey Ohm's law, while averaged diode waveforms behave as dependent power source
- Can simply replace transistor and diode with the averaged model as follows:



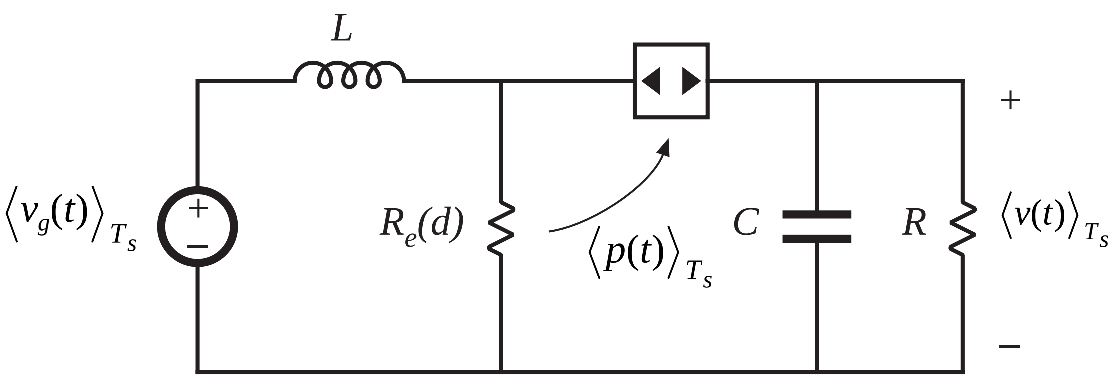
DCM buck, boost

$$R_e = \frac{2L}{d^2 T_s}$$

Buck



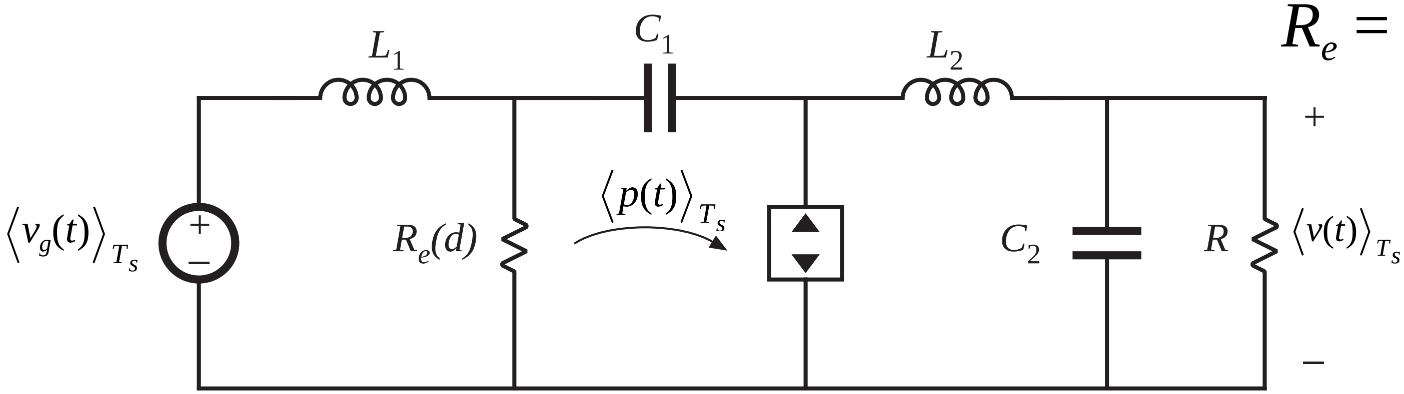
Boost



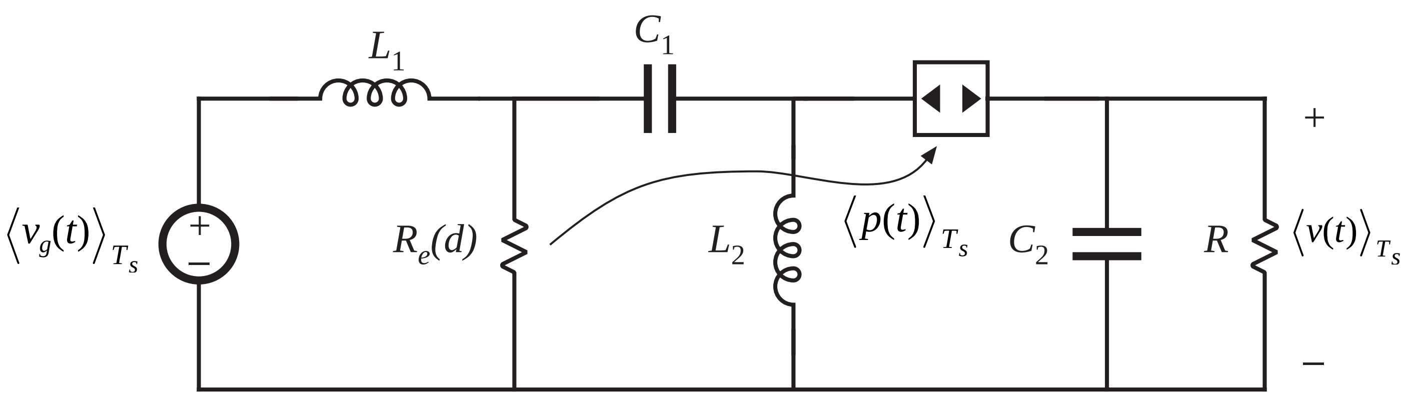
DCM Cuk, SEPIC

$$R_e = \frac{2\left(L_1 \| L_2\right)}{d^2 T_s}$$

Cuk



SEPIC

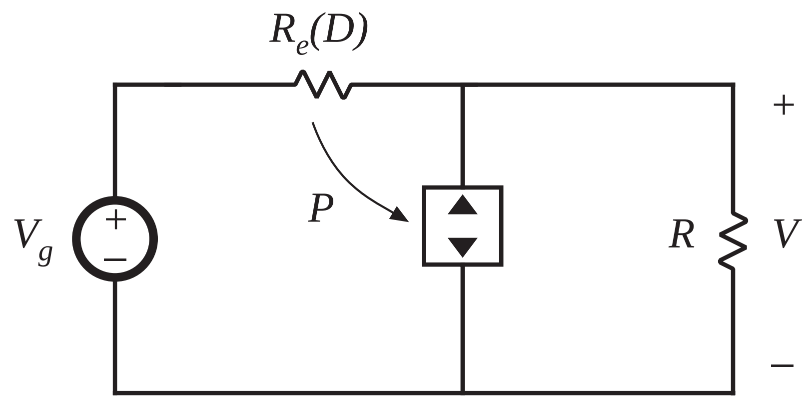


Steady-state solution: DCM buck, boost

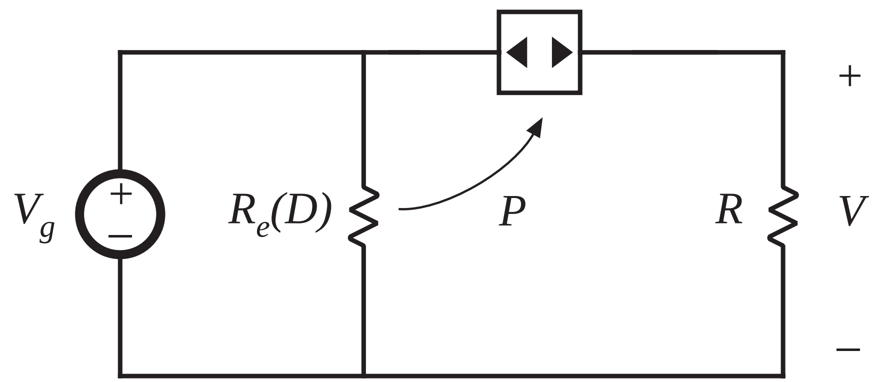
Let $L \rightarrow$ short circuit

$C \rightarrow$ open circuit

Buck



Boost



Steady-state solution of DCM / LFR models

Table 10.1. CCM and DCM conversion ratios of basic converters

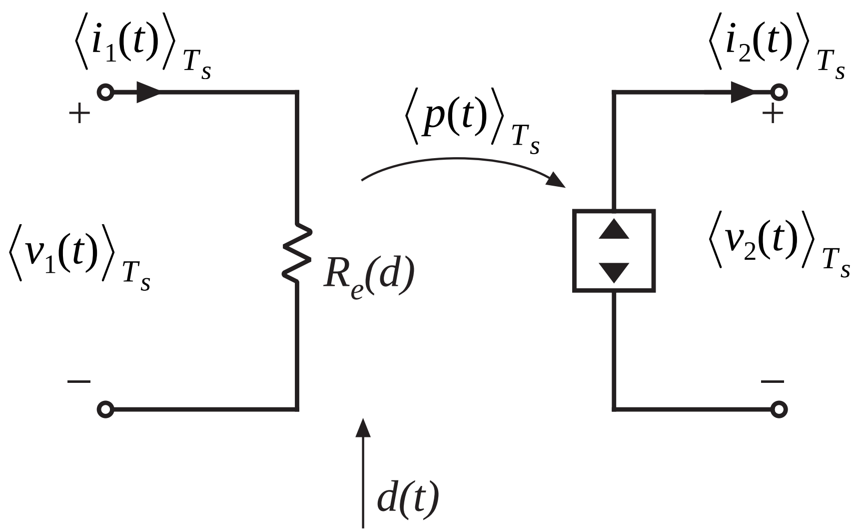
Converter	M, CCM	M, DCM
Buck	D	$\dfrac{2}{1+\sqrt{1+4R_e/R}}$
Boost	$\dfrac{1}{1-D}$	$\dfrac{1+\sqrt{1+4R/R_e}}{2}$
Buck-boost, Cuk	$\dfrac{-D}{1-D}$	$-\sqrt{\dfrac{R}{R_e}}$
SEPIC	$\dfrac{D}{1-D}$	$\sqrt{\dfrac{R}{R_e}}$

$I > I_{crit}$ for CCM
$I < I_{crit}$ for DCM

$$I_{crit} = \frac{1-D}{D}\,\frac{V_g}{R_e(D)}$$

10.2 Small-signal ac modeling of the DCM switch network

Large-signal averaged model



$$\left\langle i_1(t) \right\rangle_{T_s} = \frac{d_1^2(t)\, T_s}{2L} \left\langle v_1(t) \right\rangle_{T_s}$$

$$\left\langle i_2(t) \right\rangle_{T_s} = \frac{d_1^2(t)\, T_s}{2L} \frac{\left\langle v_1(t) \right\rangle_{T_s}^2}{\left\langle v_2(t) \right\rangle_{T_s}}$$

Perturb and linearize: let

$$d(t) = D + \hat{d}(t)$$

$$\left\langle v_1(t) \right\rangle_{T_s} = V_1 + \hat{v}_1(t)$$

$$\left\langle i_1(t) \right\rangle_{T_s} = I_1 + \hat{i}_1(t)$$

$$\left\langle v_2(t) \right\rangle_{T_s} = V_2 + \hat{v}_2(t)$$

$$\left\langle i_2(t) \right\rangle_{T_s} = I_2 + \hat{i}_2(t)$$

$$\hat{i}_1 = \frac{\hat{v}_1}{r_1} + j_1 \hat{d} + g_1 \hat{v}_2$$

$$\hat{i}_2 = -\frac{\hat{v}_2}{r_2} + j_2 \hat{d} + g_2 \hat{v}_1$$

Linearization via Taylor series

Given the nonlinear equation

$$\left\langle i_1(t) \right\rangle_{T_s} = \frac{\left\langle v_1(t) \right\rangle_{T_s}}{R_e(d(t))} = f_1\left(\left\langle v_1(t) \right\rangle_{T_s}, \left\langle v_2(t) \right\rangle_{T_s}, d(t) \right)$$

Expand in three-dimensional Taylor series about the quiescent operating point:

(for simple notation, drop angle brackets)

$$I_1 + \hat{i}_1(t) = f_1\left(V_1, V_2, D\right) + \hat{v}_1(t) \left. \frac{d f_1\left(v_1, V_2, D\right)}{d v_1} \right|_{v_1 = V_1}$$

$$+ \hat{v}_2(t) \left. \frac{d f_1\left(V_1, v_2, D\right)}{d v_2} \right|_{v_2 = V_2} + \hat{d}(t) \left. \frac{d f_1\left(V_1, V_2, d\right)}{d d} \right|_{d = D}$$

$$+ \text{higher–order nonlinear terms}$$

Equate dc and first-order ac terms

AC

$$\hat{i}_1(t) = \hat{v}_1(t)\,\frac{1}{r_1} + \hat{v}_2(t)\,g_1 + \hat{d}(t)\,j_1$$

$$\frac{1}{r_1} = \left.\frac{d\,f_1\left(v_1, V_2, D\right)}{dv_1}\right|_{v_1 = V_1} = \frac{1}{R_e(D)}$$

$$g_1 = \left.\frac{d\,f_1\left(V_1, v_2, D\right)}{dv_2}\right|_{v_2 = V_2} = 0$$

$$j_1 = \left.\frac{d\,f_1\left(V_1, V_2, d\right)}{dd}\right|_{d = D} = -\frac{V_1}{R_e^2(D)}\left.\frac{dR_e(d)}{dd}\right|_{d = D}$$

$$= \frac{2V_1}{DR_e(D)}$$

DC

$$I_1 = f_1\left(V_1, V_2, D\right) = \frac{V_1}{R_e(D)}$$

Output port
same approach

$$\left\langle i_2(t)\right\rangle_{T_s} = \frac{\left\langle v_1(t)\right\rangle_{T_s}^2}{R_e(d(t))\left\langle v_2(t)\right\rangle_{T_s}} = f_2\left(\left\langle v_1(t)\right\rangle_{T_s}, \left\langle v_2(t)\right\rangle_{T_s}, d(t)\right)$$

$$I_2 = f_2\left(V_1, V_2, D\right) = \frac{V_1^2}{R_e(D)\, V_2}$$

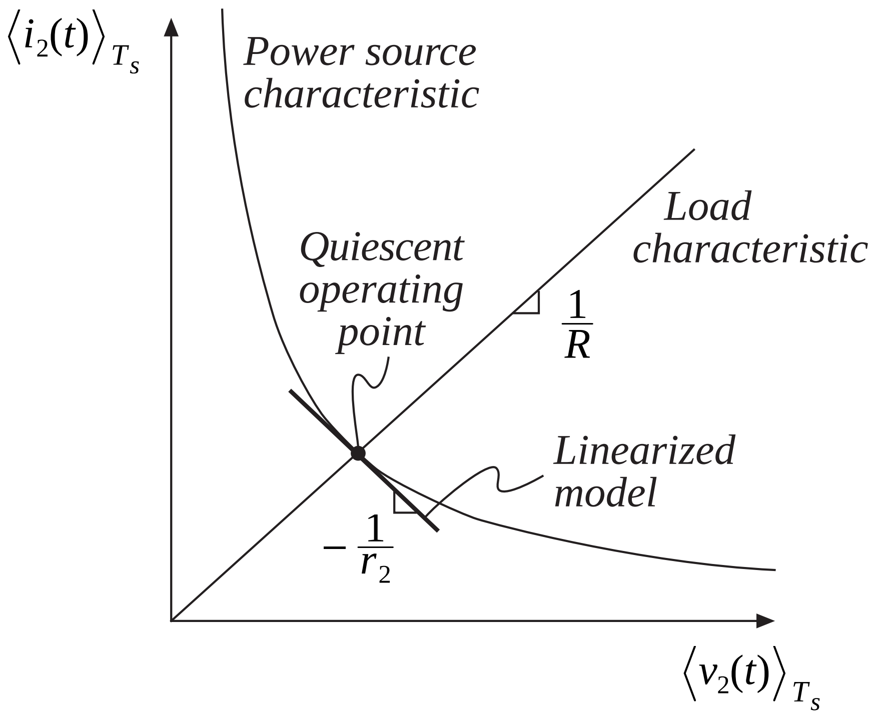
DC terms

$$\hat{i}_2(t) = \hat{v}_2(t)\left(-\frac{1}{r_2}\right) + \hat{v}_1(t) g_2 + \hat{d}(t) j_2$$

Small-signal ac linearization

Output resistance parameter r_2

$$\frac{1}{r_2} = -\left.\frac{d\,f_2\big(V_1,\,v_2,\,D\big)}{d v_2}\right|_{v_2 = V_2} = \frac{1}{R} = \frac{1}{M^2\,R_e(D)}$$



Small-signal DCM switch model parameters

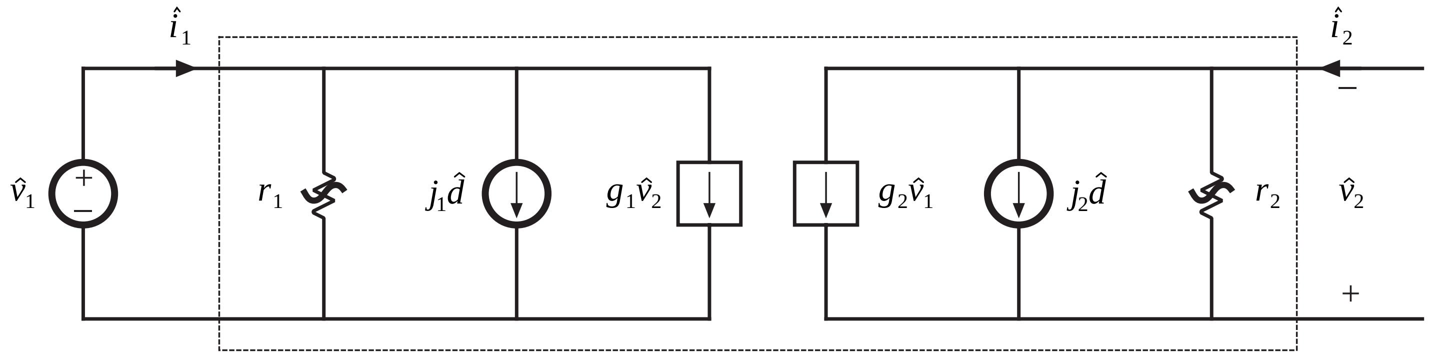
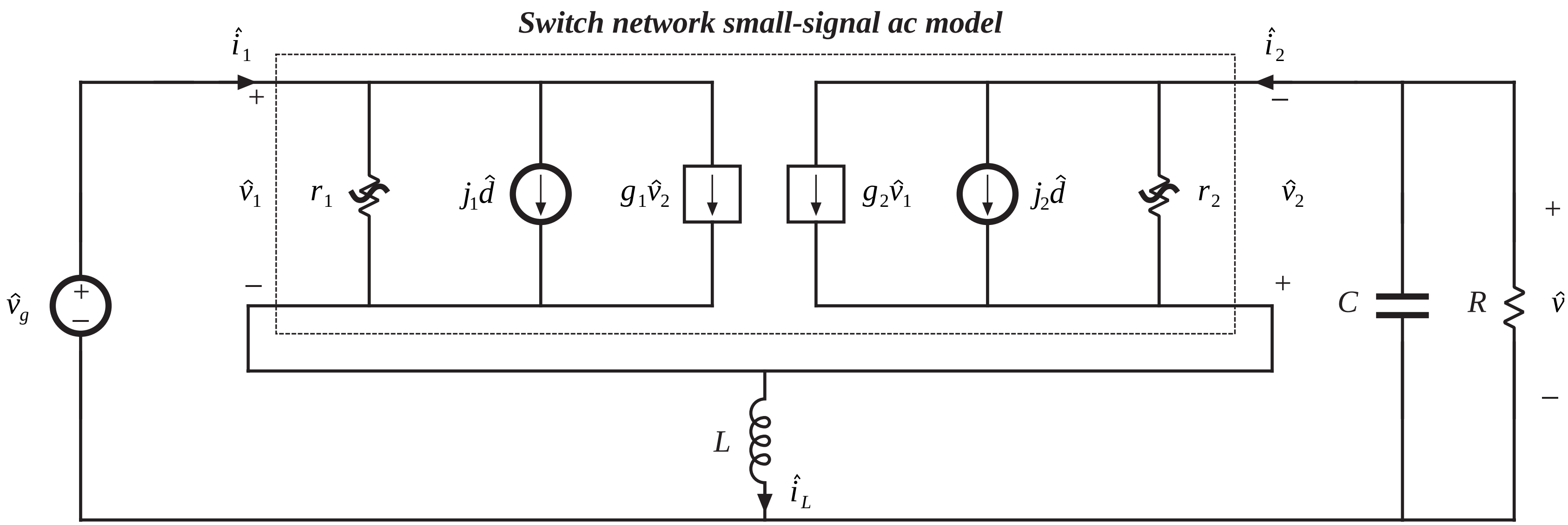


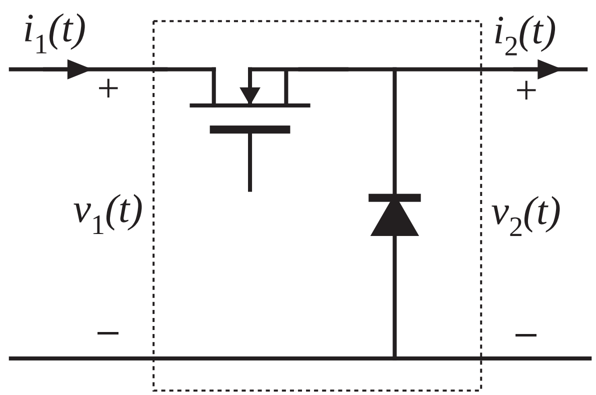
Table 10.2. Small-signal DCM switch model parameters

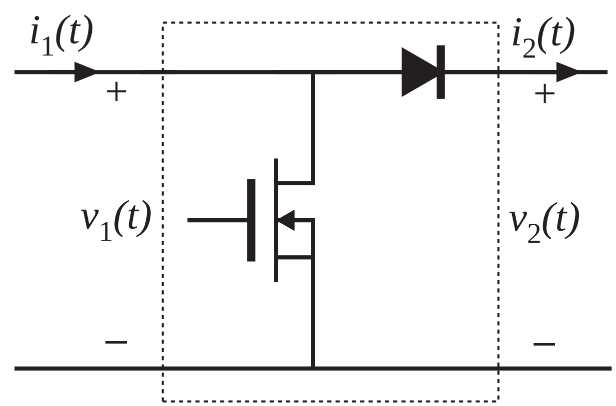
Switch type	g_1	j_1	r_1	g_2	j_2	r_2
Buck, Fig. 10.16(a)	$\frac{1}{R_e}$	$\frac{2(1-M)V_1}{DR_e}$	R_e	$\frac{2-M}{MR_e}$	$\frac{2(1-M)V_1}{DMR_e}$	M^2R_e
Boost, Fig. 10.16(b)	$\frac{1}{(M-1)^2\,R_e}$	$\frac{2MV_1}{D(M-1)R_e}$	$\frac{(M-1)^2}{M}\,R_e$	$\frac{2M-1}{(M-1)^2\,R_e}$	$\frac{2V_1}{D(M-1)R_e}$	$(M-1)^2R_e$
Buck-boost, Fig. 10.7(b)	0	$\frac{2V_1}{DR_e}$	R_e	$\frac{2M}{R_e}$	$\frac{2V_1}{DMR_e}$	M^2R_e

Small-signal ac model, DCM buck-boost example

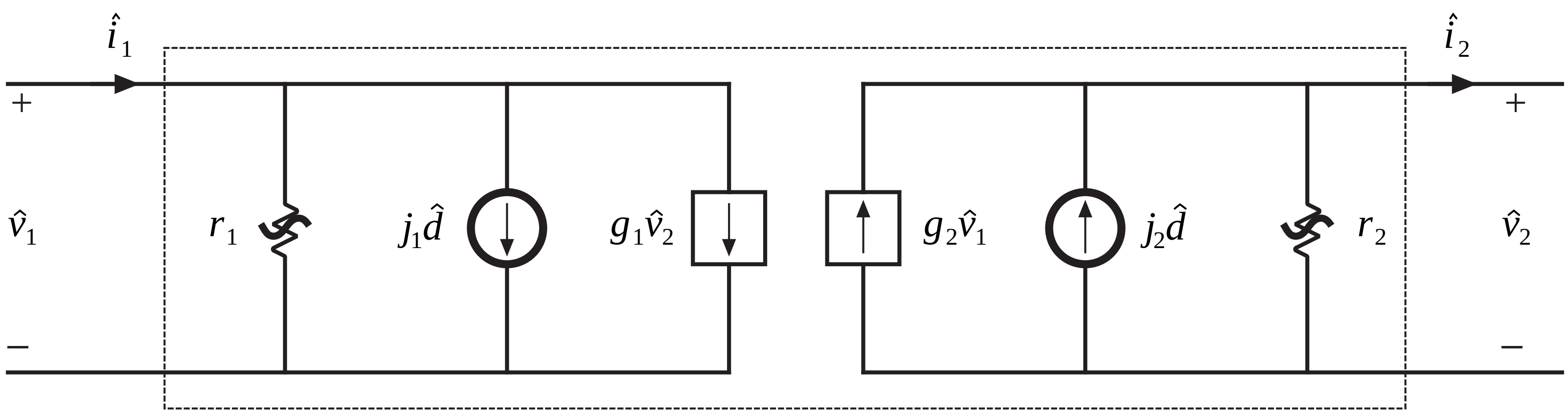


A more convenient way to model the buck and boost small-signal DCM switch networks

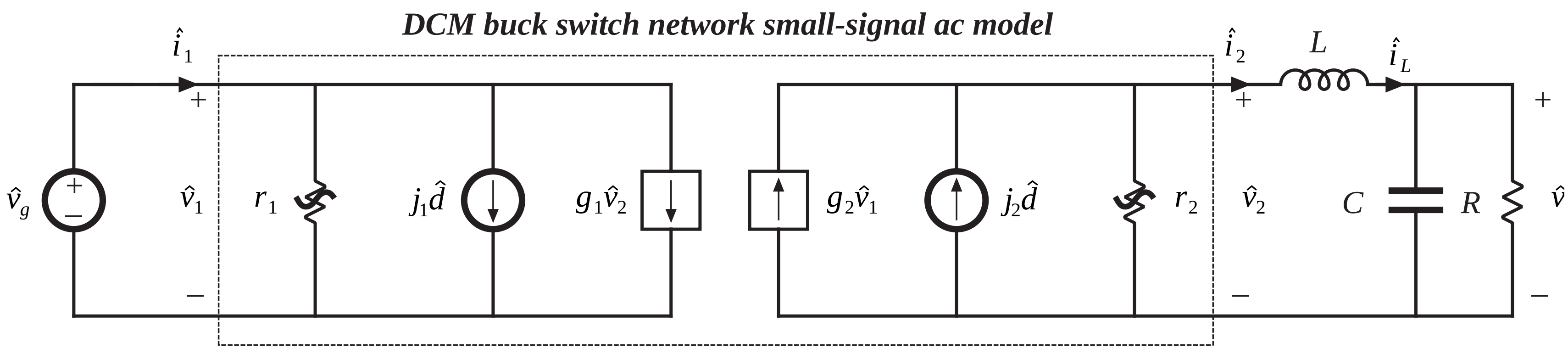


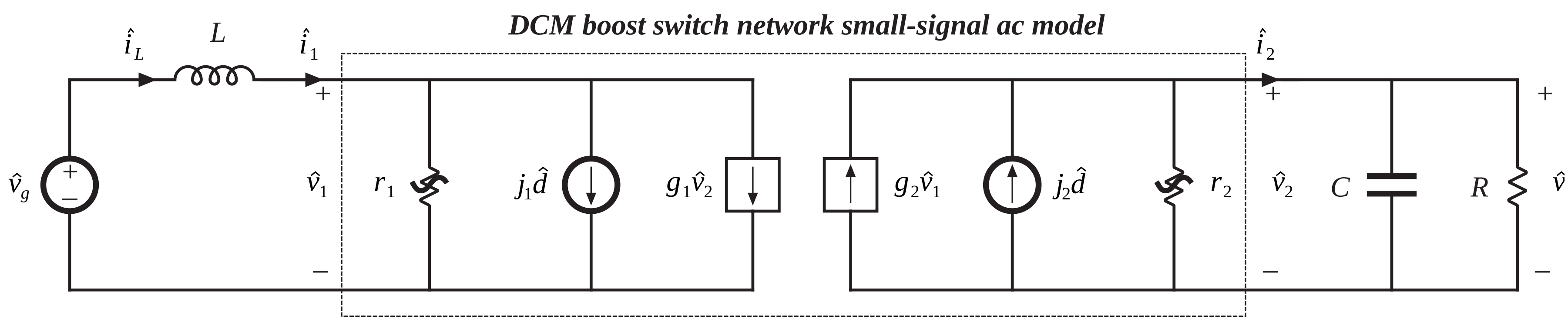


In any event, a small-signal two-port model is used, of the form



Small-signal ac models of the DCM buck and boost converters (more convenient forms)





DCM small-signal transfer functions

- When expressed in terms of R, L, C, and M (not D), the small-signal transfer functions are the same in DCM as in CCM
- Hence, DCM boost and buck-boost converters exhibit two poles and one RHP zero in control-to-output transfer functions
- ***But***, value of L is small in DCM. Hence

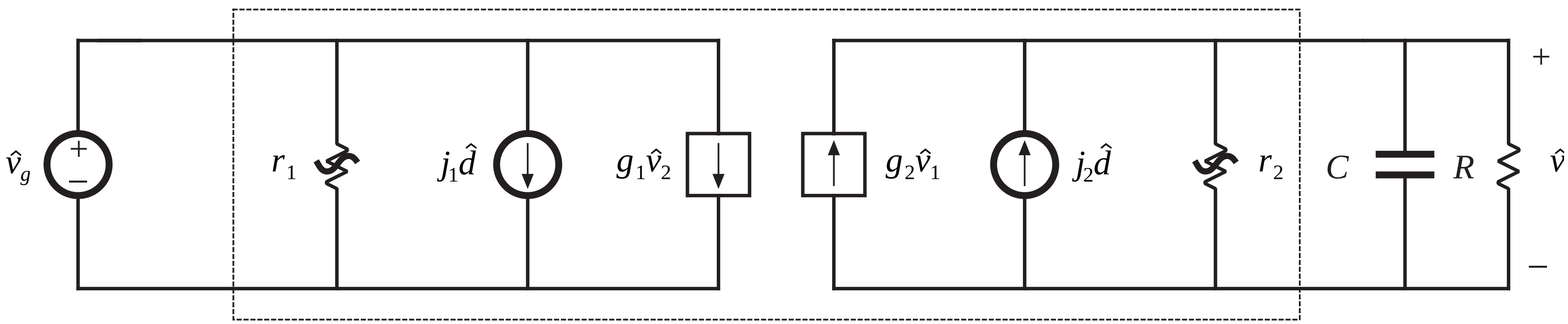
  RHP zero appears at high frequency, usually greater than switching frequency

  Pole due to inductor dynamics appears at high frequency, near to or greater than switching frequency

  So DCM buck, boost, and buck-boost converters exhibit essentially a single-pole response
- A simple approximation: let $L \to 0$

The simple approximation $L \to 0$

Buck, boost, and buck-boost converter models all reduce to



Transfer functions

control-to-output

$$G_{vd}(s) = \left.\frac{\hat{v}}{\hat{d}}\right|_{\hat{v}_g = 0} = \frac{G_{d0}}{1 + \frac{s}{\omega_p}}$$

line-to-output

$$G_{vg}(s) = \left.\frac{\hat{v}}{\hat{v}_g}\right|_{\hat{d} = 0} = \frac{G_{g0}}{1 + \frac{s}{\omega_p}}$$

with

$$G_{d0} = j_2 \left(R \,\|\, r_2\right)$$

$$\omega_p = \frac{1}{\left(R \,\|\, r_2\right) C}$$

$$G_{g0} = g_2 \left(R \,\|\, r_2\right) = M$$

Transfer function salient features

Table 10.3. Salient features of DCM converter small-signal transfer functions

Converter	G_{d0}	G_{g0}	ω_p
Buck	$\frac{2V}{D}\frac{1-M}{2-M}$	M	$\frac{2-M}{(1-M)RC}$
Boost	$\frac{2V}{D}\frac{M-1}{2M-1}$	M	$\frac{2M-1}{(M-1)RC}$
Buck-boost	$\frac{V}{D}$	M	$\frac{2}{RC}$

DCM boost example

R = 12 Ω

L = 5 μH

C = 470 μF

f_s = 100 kHz

The output voltage is regulated to be V = 36 V. It is desired to determine $G_{vd}(s)$ at the operating point where the load current is I = 3 A and the dc input voltage is V_g = 24 V.

Evaluate simple model parameters

$$P = I\left(V - V_g\right) = \left(3\text{ A}\right)\left(36\text{ V} - 24\text{ V}\right) = 36\text{ W}$$

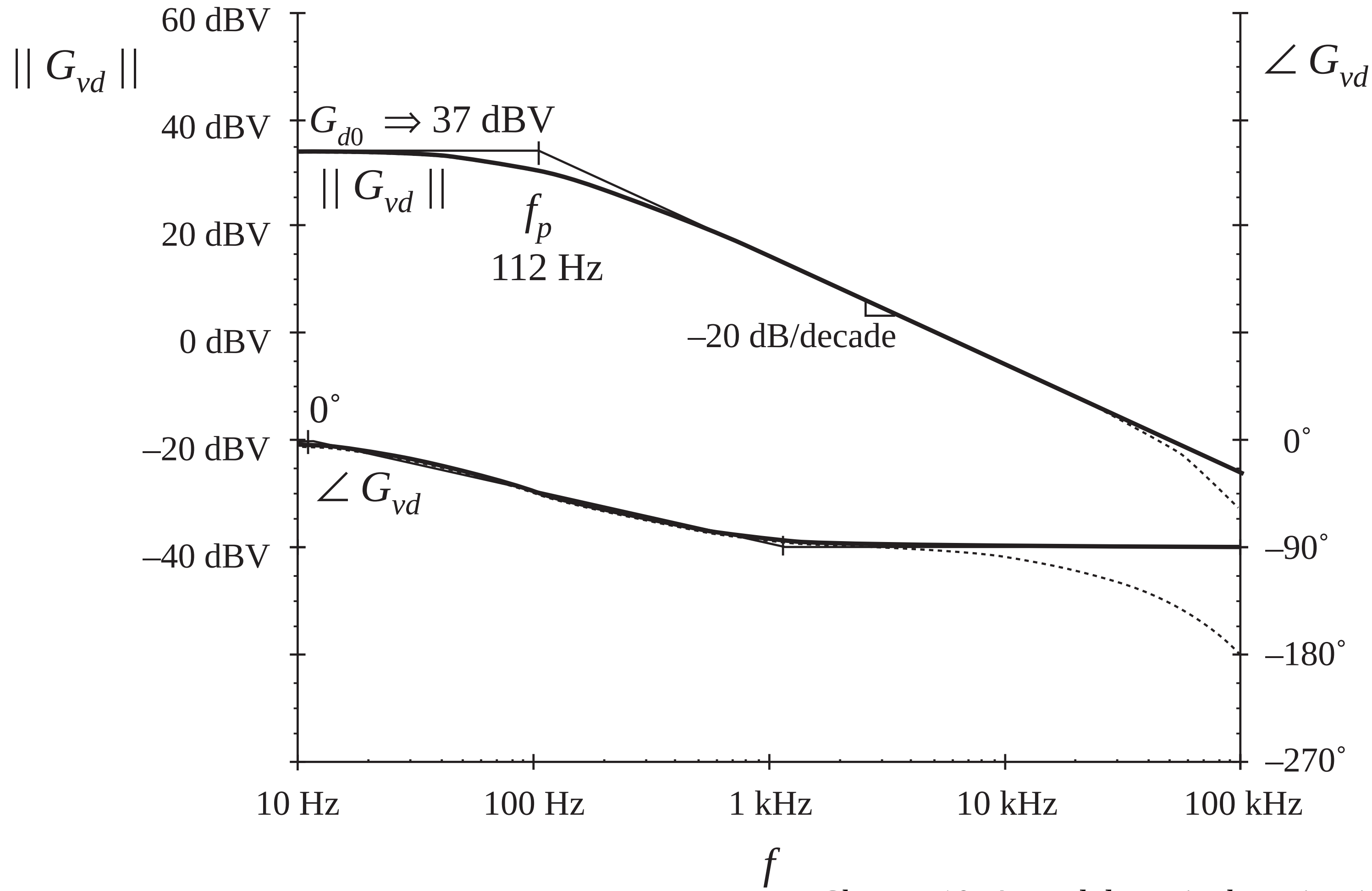
$$R_e = \frac{V_g^2}{P} = \frac{(24\text{ V})^2}{36\text{ W}} = 16\ \Omega$$

$$D = \sqrt{\frac{2L}{R_e T_s}} = \sqrt{\frac{2(5\ \mu\text{H})}{(16\ \Omega)(10\ \mu\text{s})}} = 0.25$$

$$G_{d0} = \frac{2V}{D}\,\frac{M-1}{2M-1} = \frac{2(36\text{ V})}{(0.25)}\,\frac{\left(\frac{(36\text{ V})}{(24\text{ V})} - 1\right)}{\left(2\,\frac{(36\text{ V})}{(24\text{ V})} - 1\right)} = 72\text{ V} \Rightarrow 37\text{ dBV}$$

$$f_p = \frac{\omega_p}{2\pi} = \frac{2M-1}{2\pi\,(M-1)RC} = \frac{\left(2\,\frac{(36\text{ V})}{(24\text{ V})} - 1\right)}{2\pi\left(\frac{(36\text{ V})}{(24\text{ V})} - 1\right)(12\ \Omega)(470\ \mu\text{F})} = 112\text{ Hz}$$

Control-to-output transfer function, boost example



10.3 Generalized Switch Averaging

An approach that directly relates the transfer functions of converters

operating in DCM, and/or

with current programmed control, and/or

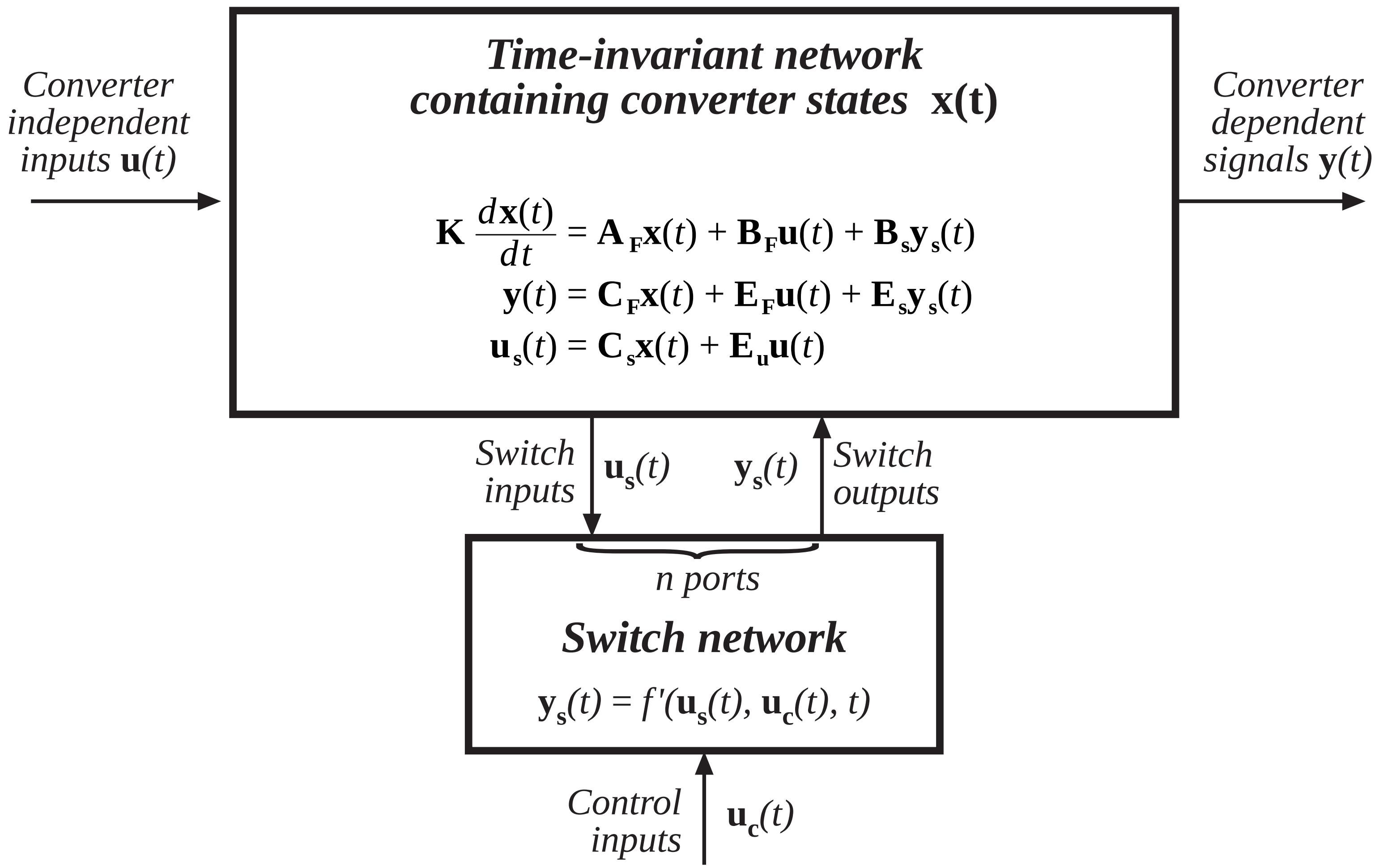
with resonant switches, and/or

with other control schemes or switch implementations,

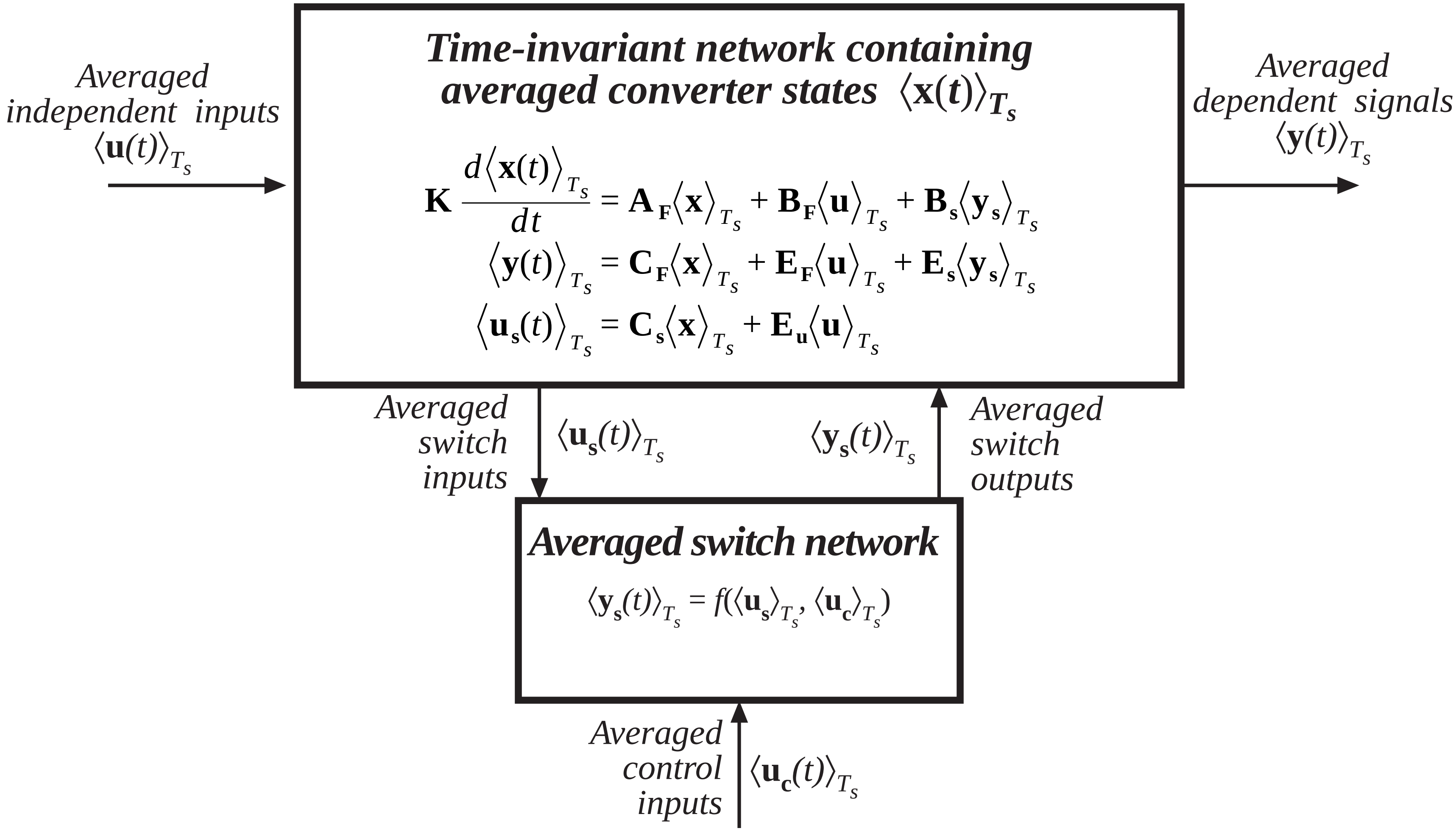
to the transfer functions of the parent CCM converter, derived in Chapter 7.

The models for these other modes, control schemes, and switch implementations are shown to be equivalent to the CCM models of Chapter 7, plus additional effective feedback loops that describe the switch behavior

Converter and switch network state equations



Averaged system equations



Averaging the switch network dependent quantities

Place switch network dependent outputs in vector $y_s(t)$, then average over one switching period, to obtain an equation of the form

$$\left\langle \mathbf{y_s}(t) \right\rangle_{T_s} = \boldsymbol{f}\left(\left\langle \mathbf{u_s}(t) \right\rangle_{T_s}, \left\langle \mathbf{u_c}(t) \right\rangle_{T_s} \right)$$

Now attempt to write the converter state equations in the same form used for CCM state-space averaging in Chapter 7. This can be done provided that the above equation can be manipulated into the form

$$\left\langle \mathbf{y_s}(t) \right\rangle_{T_s} = \mu(t)\, \mathbf{y_{s1}}(t) + \mu'(t)\, \mathbf{y_{s2}}(t)$$

where $y_{s1}(t)$ is the value of $y_s(t)$ in the CCM converter during subinterval 1

$y_{s2}(t)$ is the value of $y_s(t)$ in the CCM converter during subinterval 2

μ is called the switch conversion ratio

$\mu' = 1 - \mu$

Switch conversion ratio μ

If it is true that

$$\left\langle \mathbf{y_s}(t) \right\rangle_{T_s} = \mu(t)\, \mathbf{y_{s1}}(t) + \mu'(t)\, \mathbf{y_{s2}}(t)$$

then CCM equations can be used directly, simply by replacing the duty cycle $d(t)$ with the switch conversion ratio $\mu(t)$:

Steady-state relations are found by replacing D with μ_0

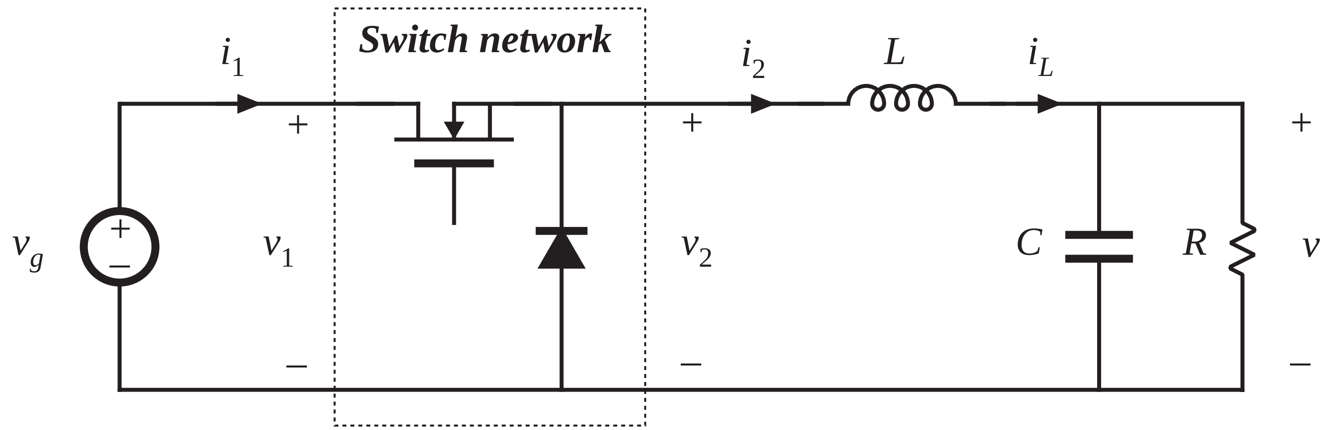
Small-signal transfer functions are found by replacing $\hat{d}(t)$ with $\hat{\mu}(t)$

The switch conversion ratio μ is a generalization of the CCM duty cycle d. In general, μ may depend on the switch independent inputs, that is, converter voltages and currents. So feedback may be built into the switch network.

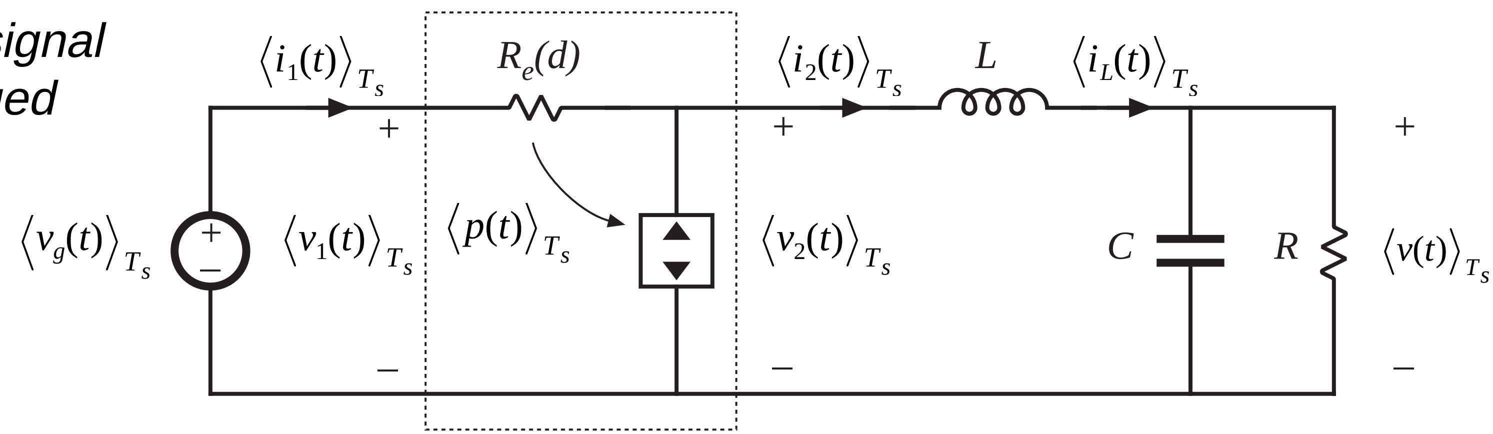
A proof follows later.

10.3.1 Buck converter example

Original converter



DCM large-signal averaged model



Defining the switch network inputs and outputs

Switch input vector

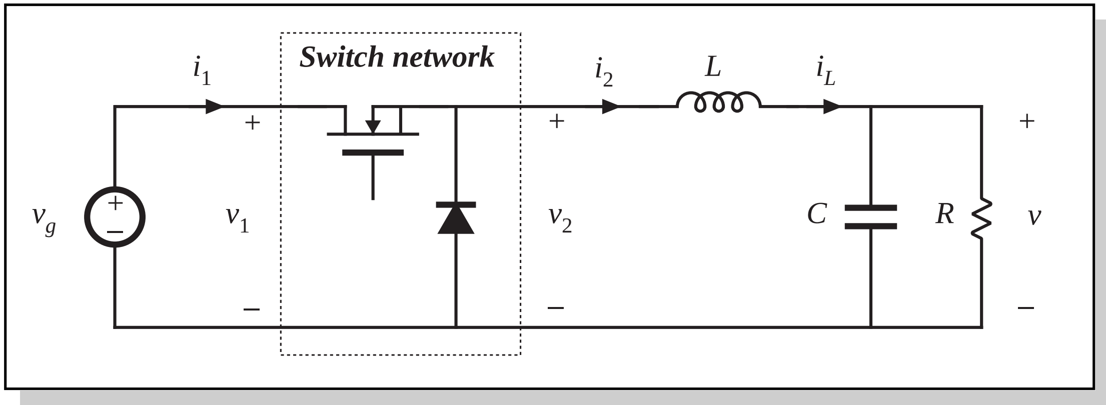
$$\mathbf{u_s}(t) = \begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}$$

Switch control input

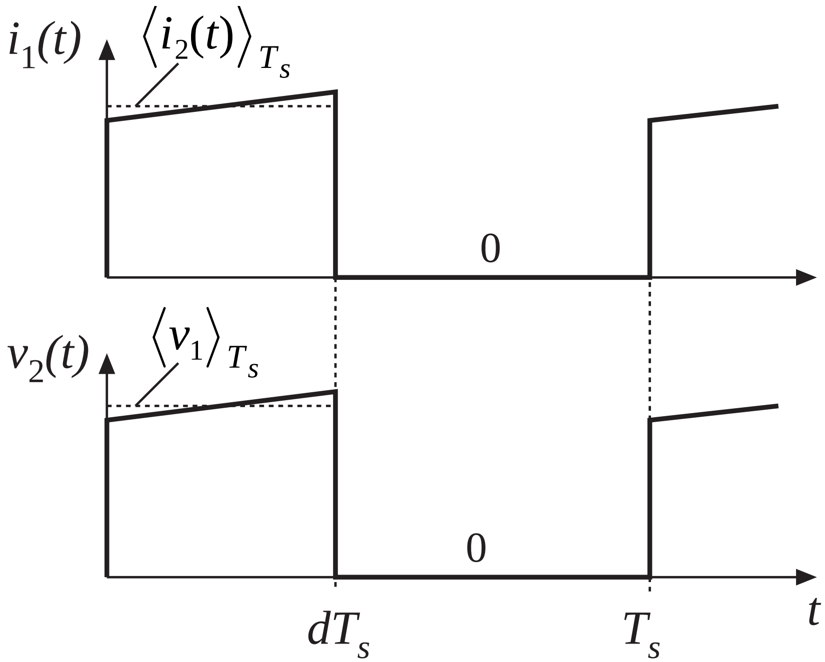
$$\mathbf{u_c}(t) = \begin{bmatrix} d(t) \end{bmatrix}$$

Switch output vector

$$\mathbf{y_s}(t) = \begin{bmatrix} v_2(t) \\ i_1(t) \end{bmatrix}$$



Switch output waveforms, CCM operation



CCM switch outputs during subintervals 1 and 2 are:

$$\mathbf{y_{s1}}(t) = \begin{bmatrix} \left\langle v_1(t) \right\rangle_{T_s} \\ \left\langle i_2(t) \right\rangle_{T_s} \end{bmatrix}, \quad \mathbf{y_{s2}}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, we should define the switch conversion ratio μ to satisfy

$$\begin{bmatrix} \left\langle v_2(t) \right\rangle_{T_s} \\ \left\langle i_1(t) \right\rangle_{T_s} \end{bmatrix} = \mu(t) \begin{bmatrix} \left\langle v_1(t) \right\rangle_{T_s} \\ \left\langle i_2(t) \right\rangle_{T_s} \end{bmatrix} + (1 - \mu(t)) \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For CCM operation, this equation is satisfied with $\mu = d$.

Solve for μ, in general

$$\left\langle \mathbf{y_s}(t) \right\rangle_{T_s} = \mu(t)\, \mathbf{y_{s1}}(t) + \mu'(t)\, \mathbf{y_{s2}}(t) \qquad \Rightarrow$$

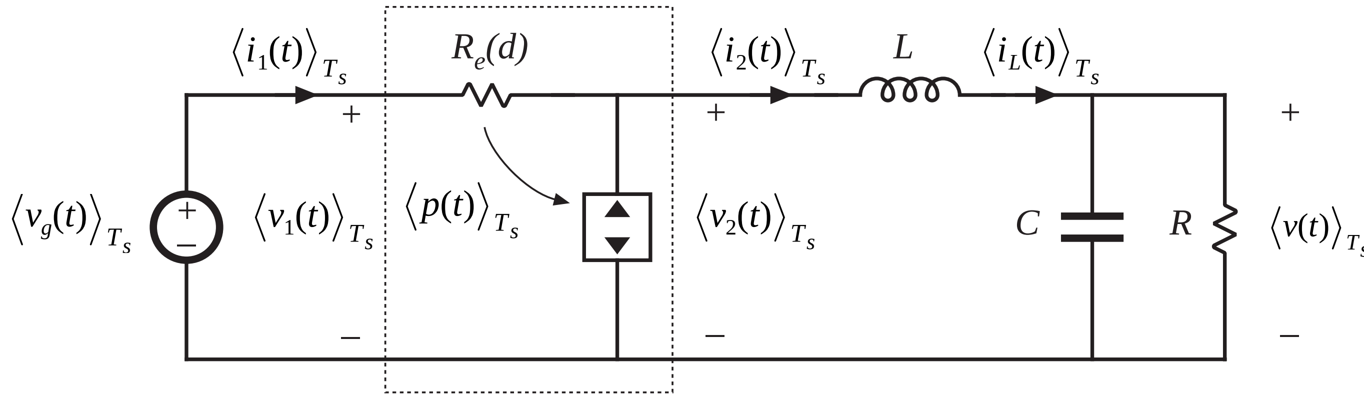
$$\begin{bmatrix} \left\langle v_2(t) \right\rangle_{T_s} \\ \left\langle i_1(t) \right\rangle_{T_s} \end{bmatrix} = \mu(t) \begin{bmatrix} \left\langle v_1(t) \right\rangle_{T_s} \\ \left\langle i_2(t) \right\rangle_{T_s} \end{bmatrix} + (1 - \mu(t)) \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Rightarrow$$

$$\mu(t) = \frac{\left\langle v_2(t) \right\rangle_{T_s}}{\left\langle v_1(t) \right\rangle_{T_s}} = \frac{\left\langle i_1(t) \right\rangle_{T_s}}{\left\langle i_2(t) \right\rangle_{T_s}}$$

This is a general definition of μ, for the switch network as defined previously for the buck converter. It is valid not only in CCM, but also in DCM and ...

Evaluation of μ

Solve averaged model for μ



$$\mu(t) = \frac{\langle v_2(t)\rangle_{T_s}}{\langle v_1(t)\rangle_{T_s}} = \frac{\langle i_1(t)\rangle_{T_s}}{\langle i_2(t)\rangle_{T_s}}$$

$$\langle v_2(t)\rangle_{T_s} = \langle v_1(t)\rangle_{T_s} - \langle i_1(t)\rangle_{T_s} R_e(d)$$

$$1 = \frac{\langle v_1(t)\rangle_{T_s}}{\langle v_2(t)\rangle_{T_s}} - \frac{\langle i_1(t)\rangle_{T_s} R_e(d)}{\langle v_2(t)\rangle_{T_s}} = \frac{1}{\mu} - \frac{\langle i_1(t)\rangle_{T_s} R_e(d)}{\langle v_2(t)\rangle_{T_s}}$$

$$\Rightarrow \quad \mu = \frac{1}{1 + R_e(d)\,\dfrac{\langle i_1(t)\rangle_{T_s}}{\langle v_2(t)\rangle_{T_s}}}$$

for DCM

Elimination of dependent quantities

we found that

$$\mu = \frac{1}{1 + R_e(d)\,\dfrac{\left\langle i_1(t)\right\rangle_{T_s}}{\left\langle v_2(t)\right\rangle_{T_s}}}$$

Lossless switch network:

$$\left\langle i_1(t)\right\rangle_{T_s}\left\langle v_1(t)\right\rangle_{T_s} = \left\langle i_2(t)\right\rangle_{T_s}\left\langle v_2(t)\right\rangle_{T_s} \qquad \Rightarrow \qquad \frac{\left\langle i_1(t)\right\rangle_{T_s}}{\left\langle v_2(t)\right\rangle_{T_s}} = \frac{\left\langle i_2(t)\right\rangle_{T_s}}{\left\langle v_1(t)\right\rangle_{T_s}}$$

Hence

$$\mu\left(\left\langle v_1(t)\right\rangle_{T_s}, \left\langle i_2(t)\right\rangle_{T_s}, d\right) = \frac{1}{1 + R_e(d)\,\dfrac{\left\langle i_2(t)\right\rangle_{T_s}}{\left\langle v_1(t)\right\rangle_{T_s}}}$$

DCM switch conversion ratio μ

$$\mu\left(\left\langle v_1(t)\right\rangle_{T_s}, \left\langle i_2(t)\right\rangle_{T_s}, d\right) = \frac{1}{1 + R_e(d)\,\dfrac{\left\langle i_2(t)\right\rangle_{T_s}}{\left\langle v_1(t)\right\rangle_{T_s}}}$$

- A general result for DCM
- Replace d of CCM expression with μ to obtain a valid DCM expression
- In DCM, switch conversion ratio is a function of not only the transistor duty cycle d, but also the switch independent terminal waveforms i_2 and v_1. The switch network contains built-in feedback.

Perturbation and linearization

$$\mu(t) = \mu_0 + \hat{\mu}(t)$$

$$\left\langle \mathbf{u_s}(t) \right\rangle_{T_s} = \mathbf{U_s} + \hat{\mathbf{u}}_\mathbf{s}(t)$$

$$\mathbf{u_c}(t) = \mathbf{U_c} + \hat{\mathbf{u}}_\mathbf{c}(t)$$

Steady-state components:

$$\mu_0 = \mu(\mathbf{U_s}, \mathbf{U_c}, D)$$

Buck example:

$$\mu_0 = \frac{1}{1 + R_e(D)\,\dfrac{I_2}{V_1}}$$

Buck example: steady-state solution

In CCM, we know that

$$\frac{V}{V_g} = M(D) = D$$

$$I_L = \frac{V}{R}$$

DCM: replace D with μ_0:

$$\frac{V}{V_g} = M(\mu_0) = \mu_0$$

$$I_2 = \frac{V}{R}$$

with $$\mu_0 = \frac{1}{1 + R_e(D)\,\frac{I_2}{V_1}}$$

Can now solve for V to obtain the usual DCM expression for V/V_g.

DCM buck example: small-signal equations

Express linearized conversion ratio as a function of switch control input and independent terminal inputs:

$$\hat{\mu}(t) = \frac{\hat{v}_1(t)}{V_s} - \frac{\hat{i}_2(t)}{I_s} + k_s\,\hat{d}(t)$$

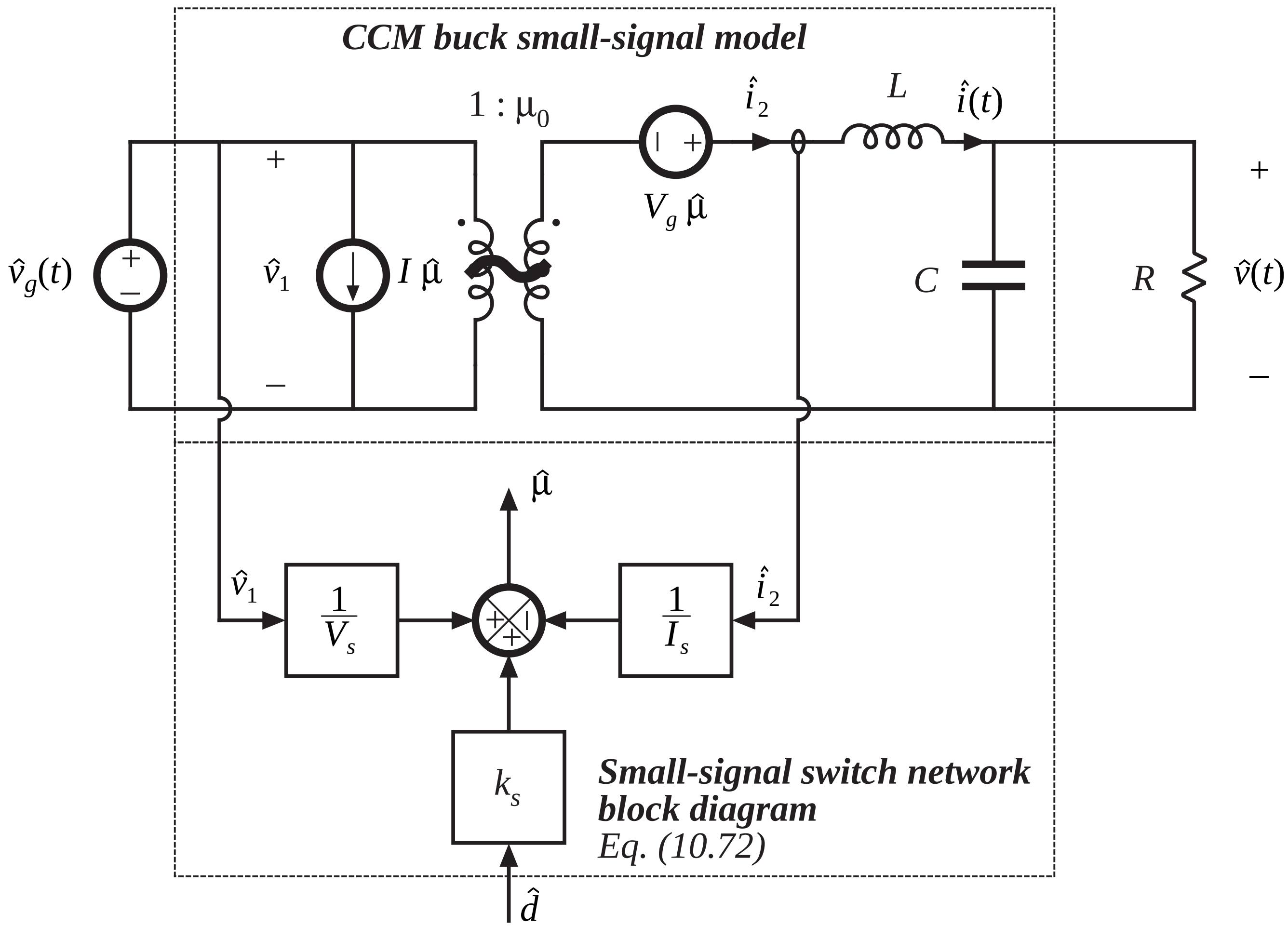
The gains are found by evaluation of derivatives at the quiescent operating point

$$\frac{1}{V_s} = \left.\frac{d\mu\left(v_1, I_2, D\right)}{dv_1}\right|_{v_1 = V_1} = \frac{\mu_0^2 I_2 R_e(D)}{V_1^2}$$

$$\frac{1}{I_s} = -\left.\frac{d\mu\left(V_1, i_2, D\right)}{di_2}\right|_{i_2 = I_2} = \frac{\mu_0^2 R_e(D)}{V_1}$$

$$k_s = \left.\frac{d\mu\left(V_1, I_2, d\right)}{dd}\right|_{d = D} = \frac{2\mu_0^2 I_2 R_e(D)}{DV_1}$$

Result: small-signal model of DCM buck



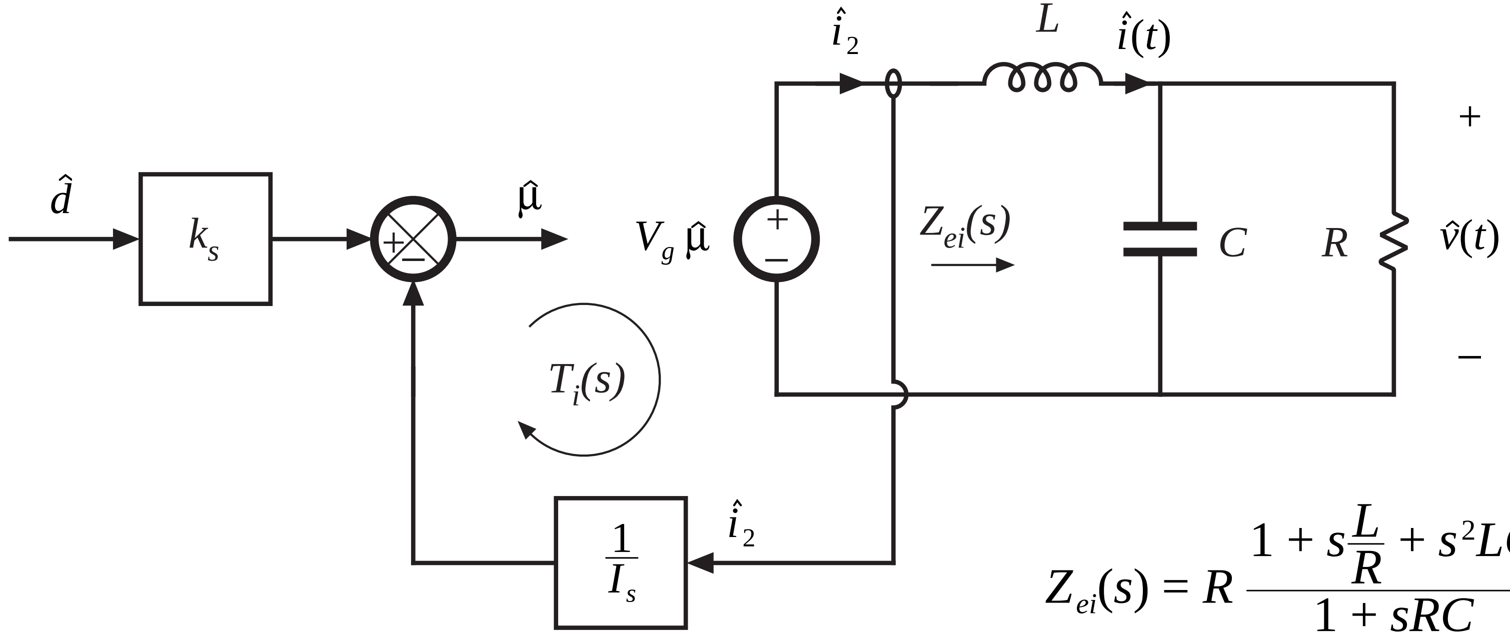
Control-to-output transfer function

$$G_{vd}(s) = \left. \frac{\hat{v}(s)}{\hat{d}(s)} \right|_{\hat{v}_g(s) = 0}$$

$$G_{vd}(s) = G_{vd\infty}(s) \frac{T_i(s)}{1 + T_i(s)}$$

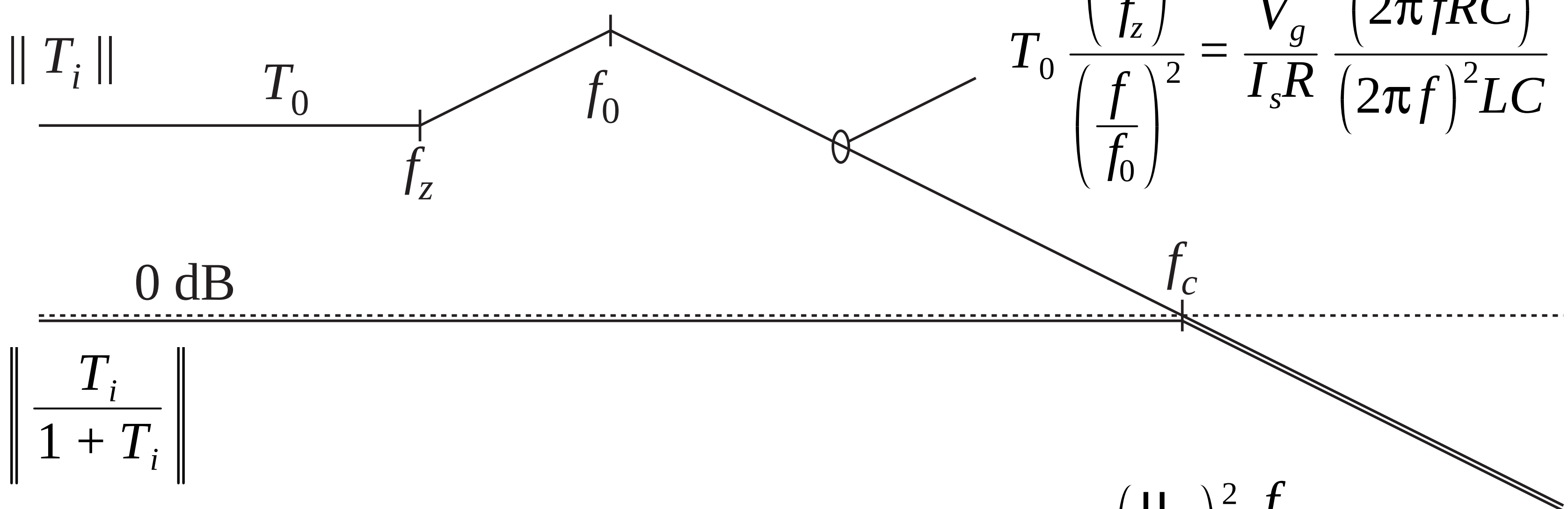
$$T_i(s) = \frac{V_g}{I_s Z_{ei}(s)}$$

$$G_{vd\infty}(s) = k_s I_s \left(R \parallel \frac{1}{sC} \right)$$



$$Z_{ei}(s) = R \frac{1 + s\frac{L}{R} + s^2 LC}{1 + sRC}$$

Magnitude of the loop gain $T_i(s)$

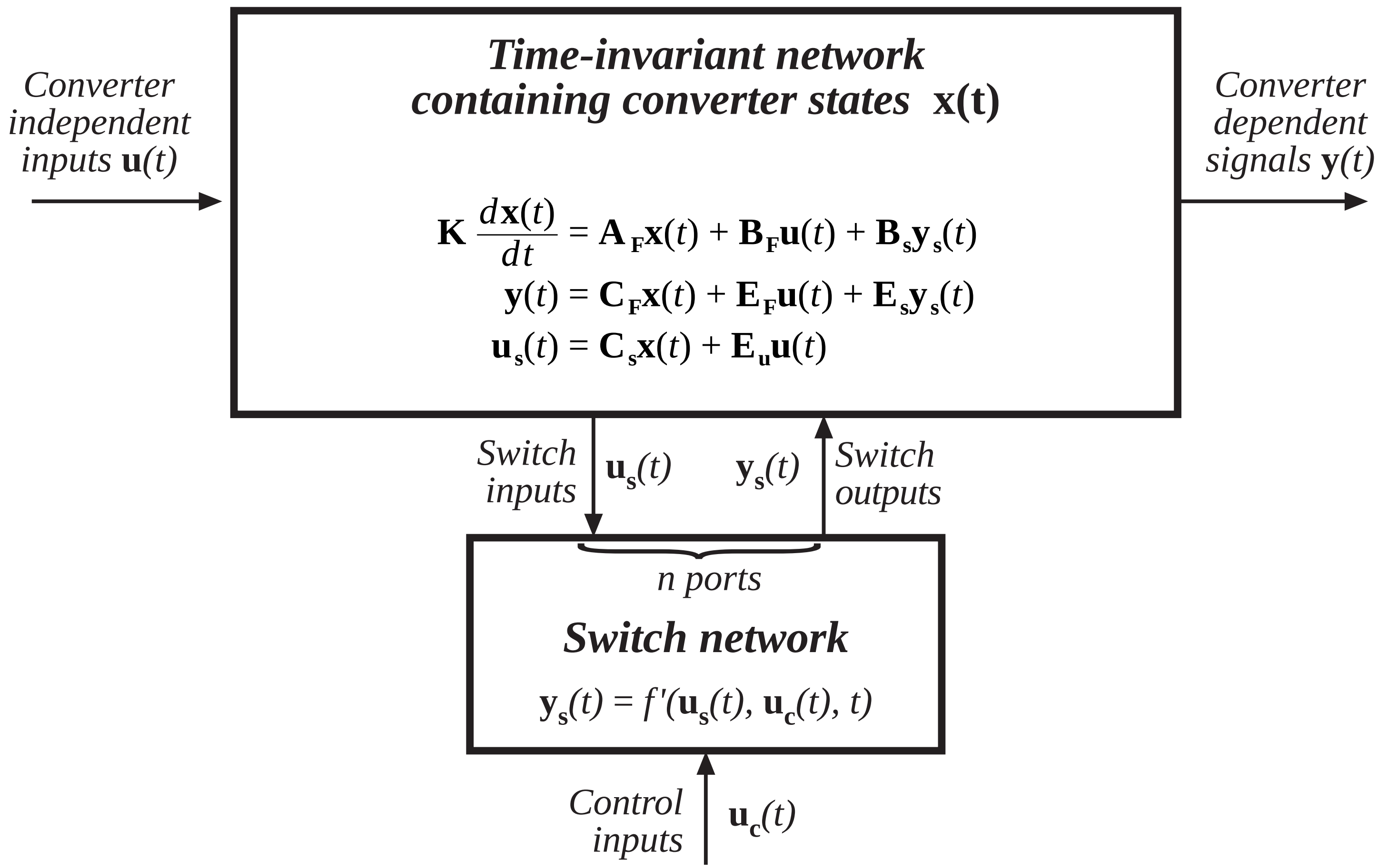


$$\left\| \frac{T_i}{1+T_i} \right\|$$

$$T_0 \frac{\left(\frac{f}{f_z}\right)}{\left(\frac{f}{f_0}\right)^2} = \frac{V_g}{I_s R} \frac{\left(2\pi f RC\right)}{\left(2\pi f\right)^2 LC}$$

$$f_c = \left(\frac{\mu_0}{D}\right)^2 \frac{f_s}{\pi}$$

10.3.2 Proof of Generalized Averaged Switch Modeling



System state equations

$$\mathbf{K}\,\frac{d\mathbf{x}(t)}{dt} = \mathbf{A_F}\mathbf{x}(t) + \mathbf{B_F}\mathbf{u}(t) + \mathbf{B_s}\mathbf{y_s}(t)$$

$$\mathbf{y}(t) = \mathbf{C_F}\mathbf{x}(t) + \mathbf{E_F}\mathbf{u}(t) + \mathbf{E_s}\mathbf{y_s}(t)$$

$$\mathbf{u_s}(t) = \mathbf{C_s}\mathbf{x}(t) + \mathbf{E_u}\mathbf{u}(t)$$

$$\mathbf{y_s}(t) = \boldsymbol{f}'\Big(\mathbf{u_s}(t),\, \mathbf{u_c}(t),\, t\Big)$$

Average:

$$\mathbf{K}\,\frac{d\left\langle \mathbf{x}(t)\right\rangle_{T_s}}{dt} = \mathbf{A_F}\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \mathbf{B_F}\left\langle \mathbf{u}(t)\right\rangle_{T_s} + \mathbf{B_s}\left\langle \mathbf{y_s}(t)\right\rangle_{T_s}$$

$$\left\langle \mathbf{y}(t)\right\rangle_{T_s} = \mathbf{C_F}\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \mathbf{E_F}\left\langle \mathbf{u}(t)\right\rangle_{T_s} + \mathbf{E_s}\left\langle \mathbf{y_s}(t)\right\rangle_{T_s}$$

$$\left\langle \mathbf{u_s}(t)\right\rangle_{T_s} = \mathbf{C_s}\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \mathbf{E_u}\left\langle \mathbf{u}(t)\right\rangle_{T_s}$$

$$\left\langle \mathbf{y_s}(t)\right\rangle_{T_s} = \boldsymbol{f}\left(\left\langle \mathbf{u_s}(t)\right\rangle_{T_s},\, \left\langle \mathbf{u_c}(t)\right\rangle_{T_s}\right)$$

Also suppose that we can write

$$\left\langle \mathbf{y_s}(t)\right\rangle_{T_s} = \mu(t)\, \mathbf{y_{s1}}(t) + \mu'(t)\, \mathbf{y_{s2}}(t)$$

Values of $\mathbf{y_s}(t)$ during subintervals 1 and 2 are defined as $\mathbf{y_{s1}}(t)$ and $\mathbf{y_{s2}}(t)$

System averaged state equations

$$\mathbf{K}\frac{d\left\langle \mathbf{x}(t)\right\rangle_{T_s}}{dt} = \mathbf{A_F}\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \mathbf{B_F}\left\langle \mathbf{u}(t)\right\rangle_{T_s} + \mu\mathbf{B_s}\mathbf{y_{s1}}(t) + \mu'\mathbf{B_s}\mathbf{y_{s2}}(t)$$

$$\left\langle \mathbf{y}(t)\right\rangle_{T_s} = \mathbf{C_F}\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \mathbf{E_F}\left\langle \mathbf{u}(t)\right\rangle_{T_s} + \mu\mathbf{E_s}\mathbf{y_{s1}}(t) + \mu'\mathbf{E_s}\mathbf{y_{s2}}(t)$$

It is desired to relate this to the result of the state-space averaging method, in which the converter state equations for subinterval 1 are written as

$$\mathbf{K}\frac{d\mathbf{x}(t)}{dt} = \mathbf{A_1}\mathbf{x}(t) + \mathbf{B_1}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C_1}\mathbf{x}(t) + \mathbf{E_1}\mathbf{u}(t)$$

with similar expressions for subinterval 2

But the time-invariant network equations predict that the converter state equations for the first subinterval are

$$\mathbf{K}\frac{d\mathbf{x}(t)}{dt} = \mathbf{A_F}\mathbf{x}(t) + \mathbf{B_F}\mathbf{u}(t) + \mathbf{B_s}\mathbf{y_{s1}}(t)$$

$$\mathbf{y}(t) = \mathbf{C_F}\mathbf{x}(t) + \mathbf{E_F}\mathbf{u}(t) + \mathbf{E_s}\mathbf{y_{s1}}(t)$$

Now equate the two expressions

Equate the state equation expressions derived via the two methods

$$\mathbf{K}\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}_1\mathbf{x}(t) + \mathbf{B}_1\mathbf{u}(t) = \mathbf{A}_F\mathbf{x}(t) + \mathbf{B}_F\mathbf{u}(t) + \mathbf{B}_s\mathbf{y}_{s1}(t)$$

$$\mathbf{y}(t) = \mathbf{C}_1\mathbf{x}(t) + \mathbf{E}_1\mathbf{u}(t) = \mathbf{C}_F\mathbf{x}(t) + \mathbf{E}_F\mathbf{u}(t) + \mathbf{E}_s\mathbf{y}_{s1}(t)$$

Solve for $\mathbf{B}_s\mathbf{y}_{s1}$ and $\mathbf{E}_s\mathbf{y}_{s1}$:

$$\mathbf{B}_s\mathbf{y}_{s1}(t) = \left(\mathbf{A}_1 - \mathbf{A}_F\right)\mathbf{x}(t) + \left(\mathbf{B}_1 - \mathbf{B}_F\right)\mathbf{u}(t)$$

$$\mathbf{E}_s\mathbf{y}_{s1}(t) = \left(\mathbf{C}_1 - \mathbf{C}_F\right)\mathbf{x}(t) + \left(\mathbf{E}_1 - \mathbf{E}_F\right)\mathbf{u}(t)$$

Result for subinterval 2:

$$\mathbf{B}_s\mathbf{y}_{s2}(t) = \left(\mathbf{A}_2 - \mathbf{A}_F\right)\mathbf{x}(t) + \left(\mathbf{B}_2 - \mathbf{B}_F\right)\mathbf{u}(t)$$

$$\mathbf{E}_s\mathbf{y}_{s2}(t) = \left(\mathbf{C}_2 - \mathbf{C}_F\right)\mathbf{x}(t) + \left(\mathbf{E}_2 - \mathbf{E}_F\right)\mathbf{u}(t)$$

Now plug these results back into averaged state equations

Averaged state equations

$$\mathbf{K}\frac{d\left\langle \mathbf{x}(t)\right\rangle_{T_s}}{dt} = \mathbf{A_F}\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \mathbf{B_F}\left\langle \mathbf{u}(t)\right\rangle_{T_s}$$

$$+ \mu\left(\left(\mathbf{A_1}-\mathbf{A_F}\right)\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \left(\mathbf{B_1}-\mathbf{B_F}\right)\left\langle \mathbf{u}(t)\right\rangle_{T_s}\right)$$

$$+ \mu'\left(\left(\mathbf{A_2}-\mathbf{A_F}\right)\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \left(\mathbf{B_2}-\mathbf{B_F}\right)\left\langle \mathbf{u}(t)\right\rangle_{T_s}\right)$$

$$\left\langle \mathbf{y}(t)\right\rangle_{T_s} = \mathbf{C_F}\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \mathbf{E_F}\left\langle \mathbf{u}(t)\right\rangle_{T_s}$$

$$+ \mu\left(\left(\mathbf{C_1}-\mathbf{C_F}\right)\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \left(\mathbf{E_1}-\mathbf{E_F}\right)\left\langle \mathbf{u}(t)\right\rangle_{T_s}\right)$$

$$+ \mu'\left(\left(\mathbf{C_2}-\mathbf{C_F}\right)\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \left(\mathbf{E_2}-\mathbf{E_F}\right)\left\langle \mathbf{u}(t)\right\rangle_{T_s}\right)$$

Collect terms

$$\mathbf{K}\,\frac{d\left\langle \mathbf{x}(t)\right\rangle_{T_s}}{dt} = \left(\mu\mathbf{A_1} + \mu'\mathbf{A_2}\right)\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \left(\mu\mathbf{B_1} + \mu'\mathbf{B_2}\right)\left\langle \mathbf{u}(t)\right\rangle_{T_s}$$

$$\left\langle \mathbf{y}(t)\right\rangle_{T_s} = \left(\mu\mathbf{C_1} + \mu'\mathbf{C_2}\right)\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \left(\mu\mathbf{E_1} + \mu'\mathbf{E_2}\right)\left\langle \mathbf{u}(t)\right\rangle_{T_s}$$

- This is the desired result. It is identical to the large-signal result of the state-space averaging method, except that the duty cycle d has been replaced with the conversion ratio μ.
- Hence, we can use any result derived via state-space averaging, by simply replacing d with μ.

Perturb and linearize

$$\mathbf{K}\frac{d\left\langle \mathbf{x}(t)\right\rangle_{T_s}}{dt} = \left(\mu\mathbf{A_1} + \mu'\mathbf{A_2}\right)\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \left(\mu\mathbf{B_1} + \mu'\mathbf{B_2}\right)\left\langle \mathbf{u}(t)\right\rangle_{T_s}$$

$$\left\langle \mathbf{y}(t)\right\rangle_{T_s} = \left(\mu\mathbf{C_1} + \mu'\mathbf{C_2}\right)\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \left(\mu\mathbf{E_1} + \mu'\mathbf{E_2}\right)\left\langle \mathbf{u}(t)\right\rangle_{T_s}$$

Let

$$\left\langle \mathbf{x}(t)\right\rangle_{T_s} = \mathbf{X} + \hat{\mathbf{x}}(t)$$

$$\left\langle \mathbf{u}(t)\right\rangle_{T_s} = \mathbf{U} + \hat{\mathbf{u}}(t)$$

$$\left\langle \mathbf{y}(t)\right\rangle_{T_s} = \mathbf{Y} + \hat{\mathbf{y}}(t)$$

$$\mu(t) = \mu_0 + \hat{\mu}(t)$$

$$\left\langle \mathbf{u_s}(t)\right\rangle_{T_s} = \mathbf{U_s} + \hat{\mathbf{u}}_\mathbf{s}(t)$$

$$\left\langle \mathbf{u_c}(t)\right\rangle_{T_s} = \mathbf{U_c} + \hat{\mathbf{u}}_\mathbf{c}(t)$$

Result

where

$$\mathbf{A} = \left(\mu_0 \mathbf{A_1} + \mu_0' \mathbf{A_2}\right)$$
$$\mathbf{B} = \left(\mu_0 \mathbf{B_1} + \mu_0' \mathbf{B_2}\right)$$
$$\mathbf{C} = \left(\mu_0 \mathbf{C_1} + \mu_0' \mathbf{C_2}\right)$$
$$\mathbf{E} = \left(\mu_0 \mathbf{E_1} + \mu_0' \mathbf{E_2}\right)$$

$$\mathbf{k_s}^T = \left. \frac{d\mu\left(\left\langle \mathbf{u_s}(t)\right\rangle_{T_s}, \left\langle \mathbf{u_c}(t)\right\rangle_{T_s}\right)}{d\left\langle \mathbf{u_s}(t)\right\rangle_{T_s}} \right|_{\substack{\left\langle \mathbf{u_s}(t)\right\rangle_{T_s} = \mathbf{U_s} \\ \left\langle \mathbf{u_c}(t)\right\rangle_{T_s} = \mathbf{U_c}}}$$

$$\mathbf{k_c}^T = \left. \frac{d\mu\left(\left\langle \mathbf{u_s}(t)\right\rangle_{T_s}, \left\langle \mathbf{u_c}(t)\right\rangle_{T_s}\right)}{d\left\langle \mathbf{u_c}(t)\right\rangle_{T_s}} \right|_{\substack{\left\langle \mathbf{u_s}(t)\right\rangle_{T_s} = \mathbf{U_s} \\ \left\langle \mathbf{u_c}(t)\right\rangle_{T_s} = \mathbf{U_c}}}$$

DC model

$$\mathbf{0} = \mathbf{AX} + \mathbf{BU}$$
$$\mathbf{Y} = \mathbf{CX} + \mathbf{EU}$$

Small-signal ac model

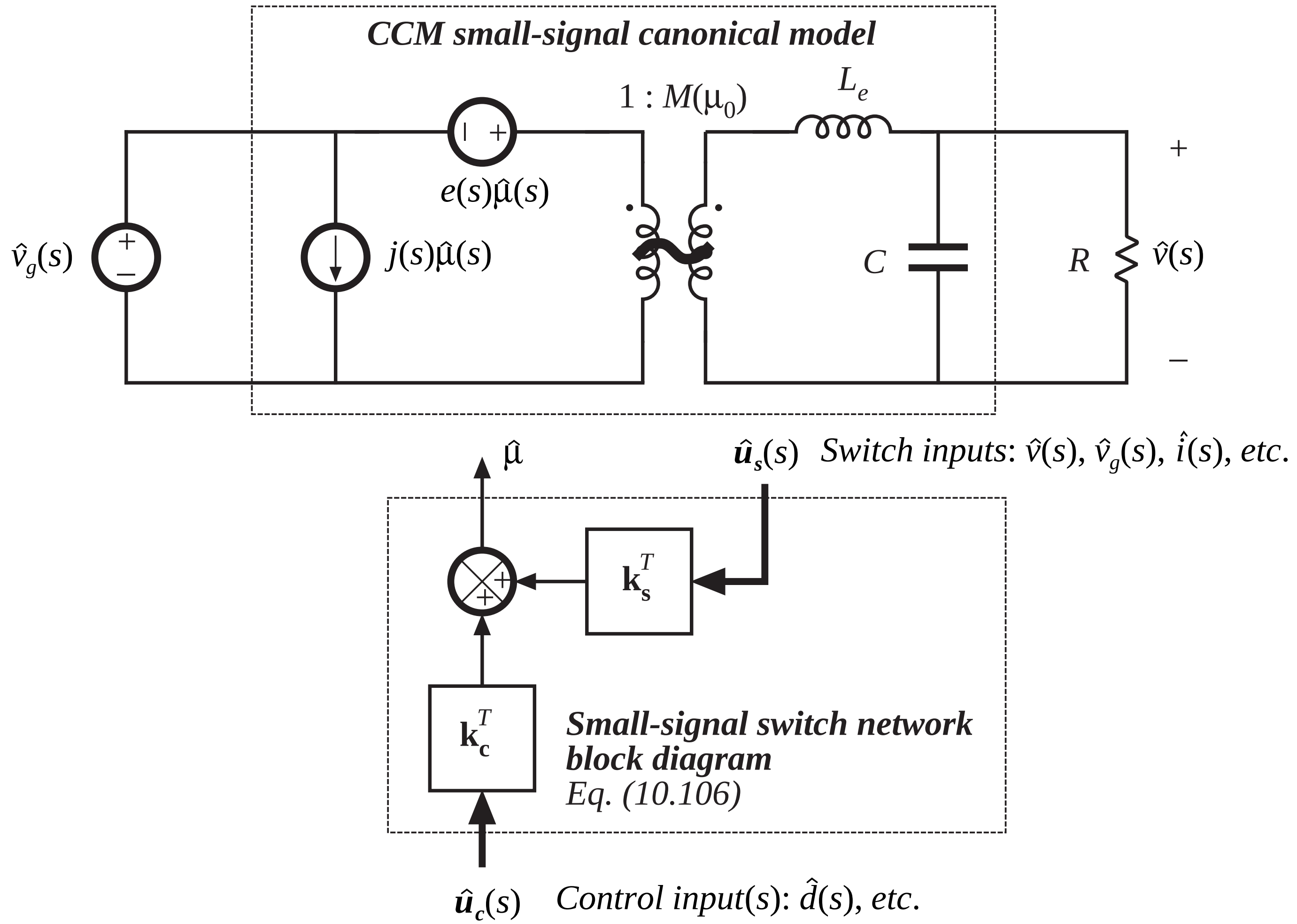
$$\mathbf{K}\frac{d\hat{\mathbf{x}}(t)}{dt} = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\hat{\mathbf{u}}(t) + \left(\left(\mathbf{A_1} - \mathbf{A_2}\right)\mathbf{X} + \left(\mathbf{B_1} - \mathbf{B_2}\right)\mathbf{U}\right)\hat{\mu}(t)$$

$$\hat{\mathbf{y}}(t) = \mathbf{C}\hat{\mathbf{x}}(t) + \mathbf{E}\hat{\mathbf{u}}(t) + \left(\left(\mathbf{C_1} - \mathbf{C_2}\right)\mathbf{X} + \left(\mathbf{E_1} - \mathbf{E_2}\right)\mathbf{U}\right)\hat{\mu}(t)$$

with the linearized switch gains

$$\hat{\mu}(t) = \mathbf{k_s}^T \hat{\mathbf{u}}_\mathbf{s}(t) + \mathbf{k_c}^T \hat{\mathbf{u}}_\mathbf{c}(t)$$

A generalized canonical model



10.4 Summary of Key Points

1. In the discontinuous conduction mode, the average transistor voltage and current are proportional, and hence obey Ohm's law. An averaged equivalent circuit can be obtained by replacing the transistor with an effective resistor $R_e(d)$. The average diode voltage and current obey a power source characteristic, with power equal to the power effectively dissipated by R_e. In the averaged equivalent circuit, the diode is replaced with a dependent power source.
2. The two-port lossless network consisting of an effective resistor and power source, which results from averaging the transistor and diode waveforms of DCM converters, is called a loss-free resistor. This network models the basic power-processing functions of DCM converters, much in the same way that the ideal dc transformer models the basic functions of CCM converters.
3. The large-signal averaged model can be solved under equilibrium conditions to determine the quiescent values of the converter currents and voltages. Average power arguments can often be used.

Key points

4. A small-signal ac model for the DCM switch network can be derived by perturbing and linearizing the loss-free resistor network. The result has the form of a two-port y-parameter model. The model describes the small-signal variations in the transistor and diode currents, as functions of variations in the duty cycle and in the transistor and diode ac voltage variations. This model is most convenient for ac analysis of the buck-boost converter.
5. To simplify the ac analysis of the DCM buck and boost converters, it is convenient to define two other forms of the small-signal switch model, corresponding to the switch networks of Figs. 10.16(a) and 10.16(b). These models are also y-parameter two-port models, but have different parameter values.

Key points

6. Since the inductor value is small when the converter operates in the discontinuous conduction mode, the inductor dynamics of the DCM buck, boost, and buck-boost converters occur at high frequency, above or just below the switching frequency. Hence, in most cases the inductor dynamics can be ignored. In the small-signal ac model, the inductance L is set to zero, and the remaining model is solved relatively easily for the low-frequency converter dynamics. The DCM buck, boost, and buck-boost converters exhibit transfer functions containing a single low-frequency dominant pole.
7. It is also possible to adapt the CCM models developed in Chapter 7 to treat converters with switches that operate in DCM, as well as other switches discussed in later chapters. The switch conversion ratio μ is a generalization of the duty cycle d of CCM switch networks; this quantity can be substituted in place of d in any CCM model. The result is a model that is valid for DCM operation. Hence, existing CCM models can be adapted directly.

Key points

8. The conversion ratio μ of DCM switch networks is a function of the applied voltage and current. As a result, the switch network contains effective feedback. So the small-signal model of a DCM converter can be expressed as the CCM converter model, plus effective feedback representing the behavior of the DCM switch network. Two effects of this feedback are increase of the converter output impedance via current feedback, and decrease of the Q-factor of the transfer function poles. The pole arising from the inductor dynamics occurs at the crossover frequency of the effective current feedback loop.