Chapter 21

Derivations for the Design Equations

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Output Power, P_o, Versus Apparent Power, P_t, Capability

Introduction

Output power, P_o, is of the greatest interest to the user. To the transformer designer, the apparent power, P_t, which is associated with the geometry of the transformer, is of greater importance. Assume, for the sake of simplicity, that the core of an isolation transformer has only two windings in the window area, a primary and a secondary. Also, assume that the window area, W_a, is divided up in proportion to the power-handling capability of the windings, using equal current density. The primary winding handles, P_{in}, and the secondary handles, P_o, to the load. Since the power transformer has to be designed to accommodate the primary, P_{in}, and, P_o, then,

By definition:

$$P_t = P_{in} + P_o, \quad \text{[watts]}$$
$$P_{in} = \frac{P_o}{\eta}, \quad \text{[watts]} \qquad \text{[21-A1]}$$

The primary turns can be expressed using Faraday's Law:

$$N_p = \frac{V_p\left(10^4\right)}{A_c B_{ac} f K_f}, \quad \text{[turns]} \qquad \text{[21-A2]}$$

The winding area of a transformer is fully utilized when:

$$K_u W_a = N_p A_{wp} + N_s A_{ws} \qquad \text{[21-A3]}$$

By definition the wire area is:

$$A_w = \frac{I}{J}, \quad \text{[cm}^2\text{]} \qquad \text{[21-A4]}$$

Rearranging the equation shows:

$$K_u W_a = N_p\left(\frac{I_p}{J}\right) + N_s\left(\frac{I_s}{J}\right) \qquad \text{[21-A5]}$$

Now, substitute in Faraday's Equation:

$$K_u W_a = \frac{V_p\left(10^4\right)}{A_c B_{ac} f K_f}\left(\frac{I_p}{J}\right) + \frac{V_s\left(10^4\right)}{A_c B_{ac} f K_f}\left(\frac{I_s}{J}\right) \quad [21\text{-A6}]$$

Rearranging shows:

$$W_a A_c = \frac{\left[\left(V_p I_p\right) + \left(V_s I_s\right)\right]\left(10^4\right)}{B_{ac} f J K_f K_u}, \quad [\text{cm}^4] \quad [21\text{-A7}]$$

The output power, P_o, is:

$$P_o = V_s I_s, \quad [\text{watts}] \quad [21\text{-A8}]$$

The input power, P_{in}, is:

$$P_{in} = V_p I_p, \quad [\text{watts}] \quad [21\text{-A9}]$$

Then:

$$P_t = P_{in} + P_o, \quad [\text{watts}] \quad [21\text{-A10}]$$

Transformer Derivation for the Core Geometry, K_g

Introduction

Although most transformers are designed for a given temperature rise, they can also be designed for a given regulation. The regulation and power-handling ability of a core are related to two constants, K_g and K_e by the equation:

$$P_t = 2K_g K_e \alpha, \quad \text{[watts]} \quad \text{[21-B1]}$$

Where:

$$\alpha = \text{Regulation}, \quad [\%]$$

The constant, K_g, is a function of the core geometry:

$$K_g = f\left(A_c, W_a, \text{MLT}\right) \quad \text{[21-B2]}$$

The constant, K_e, is a function of the magnetic and electrical operating conditions:

$$K_e = g\left(f, B_m\right) \quad \text{[21-B3]}$$

The derivation of the specific functions for, K_g and K_e, is as follows: First, assume there is a two-winding transformer with equal primary and secondary regulation, as schematically shown in Figure 21-B1. The primary winding has a resistance of, R_p, ohms, and the secondary winding has a resistance of, R_s ohms:

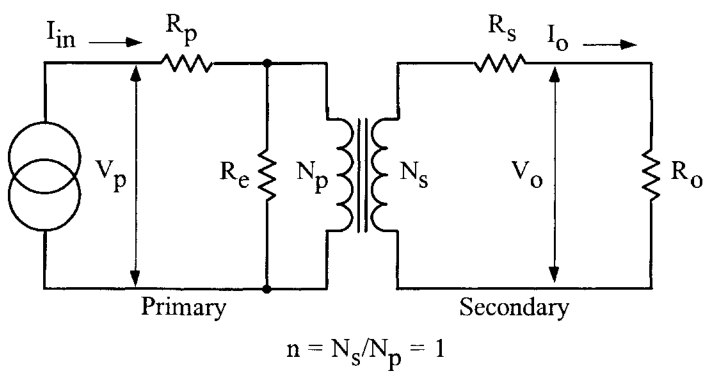

Figure 21-B1. Isolation Transformer.

$$\alpha = \frac{\Delta V_p}{V_p}(100) + \frac{\Delta V_s}{V_s}(100) \quad \text{[21-B4]}$$

The assumption, for simplicity, is that R_e is infinity (no core loss).

And:

$$I_{in} = I_o \quad [21\text{-B5}]$$

Then:

$$\Delta V_p = I_p R_p = \Delta V_s = I_s R_s, \quad \text{[volts]} \quad [21\text{-B6}]$$

$$\alpha = 2\frac{I_p R_p}{V_p}(100) \quad [21\text{-B7}]$$

Multiply the numerator and denominator by V_p:

$$\alpha = 200\frac{I_p R_p}{V_p}\left(\frac{V_p}{V_p}\right) \quad [21\text{-B8}]$$

$$\alpha = 200\frac{R_p VA}{V_p^2} \quad [21\text{-B9}]$$

From the resistivity formula, it is easily shown that:

$$R_p = \frac{(\text{MLT}) N_p^2}{W_a K_p}\rho \quad [21\text{-B10}]$$

Where:

$\rho = 1.724\ (10^{-6})$ ohm cm
K_p is the window utilization factor (primary)
K_s is the window utilization factor (secondary)

$$K_p = \frac{K_u}{2} = K_s \quad [21\text{-B11}]$$

Faraday's Law expressed in metric units is:

$$V_p = K_f f N_p A_c B_m \left(10^{-4}\right) \quad [21\text{-B12}]$$

Where:

K_f = 4.0 for a square wave.
K_f = 4.44 for a sine wave.

Substituting Equation 21-B10 and 21-B12, for R_p and V_p, in Equation [21-B13]:

$$VA = \frac{E_p^2}{200 R_p}\alpha \quad [21\text{-B13}]$$

The primary VA is:

$$VA = \frac{\left(K_f f N_p A_c B_m \left(10^{-4}\right)\right)\left(K_f f N_p A_c B_m \left(10^{-4}\right)\right)}{200\left(\frac{(\text{MLT}) N_p^2}{W_a K_p} \rho\right)} \alpha \quad [21\text{-B14}]$$

Simplify:

$$VA = \frac{K_f^2 f^2 A_c^2 B_m^2 W_a K_p \left(10^{-10}\right)}{2(\text{MLT})\rho} \alpha \quad [21\text{-B15}]$$

Inserting $1.724(10^{-6})$ for ρ:

$$VA = \frac{0.29 K_f^2 f^2 A_c^2 B_m^2 W_a K_p \left(10^{-4}\right)}{\text{MLT}} \alpha \quad [21\text{-B16}]$$

Let primary electrical equal:

$$K_e = 0.29 K_f^2 f^2 B_m^2 \left(10^{-4}\right) \quad [21\text{-B17}]$$

Let the primary core geometry equal:

$$K_g = \frac{W_a A_c^2 K_p}{\text{MLT}}, \quad [\text{cm}^5] \quad [21\text{-B18}]$$

The total transformer window utilization factor is:

$$\begin{aligned} K_p + K_s &= K_u \\ K_p &= \frac{K_u}{2} = K_s \end{aligned} \quad [21\text{-B19}]$$

When this value for K_p is put into Equation [21-B16], then:

$$VA = K_e K_g \alpha \quad [21\text{-B20}]$$

Where:

$$K_e = 0.145 K_f^2 f^2 B_m^2 \left(10^{-4}\right) \quad [21\text{-B21}]$$

The above VA is the primary power, and the window utilization factor, K_u, includes both the primary and secondary coils.

$$K_g = \frac{W_a A_c^2 K_p}{\text{MLT}}, \quad [\text{cm}^5] \quad [21\text{-B22}]$$

Regulation of a transformer is related to the copper loss, as shown in Equation [21-B23]:

$$\alpha = \frac{P_{cu}}{P_o}(100), \quad [\%] \quad [21\text{-B23}]$$

The total VA of the transformer is primary plus secondary:

$$\begin{aligned} &\text{Primary,} \quad VA = K_e K_g \alpha \\ &\text{plus} \qquad\qquad\qquad\qquad [21\text{-B24}] \\ &\text{Secondary,} \quad VA = K_e K_g \alpha \end{aligned}$$

The apparent power, P_t, then is:

$$\begin{aligned} P_t &= (\text{Primary}) K_e K_g \alpha + (\text{Secondary}) K_e K_g \alpha \\ P_t &= 2 K_e K_g \alpha \end{aligned} \quad [21\text{-B25}]$$

Transformer Derivation for the Area Product, A_p

Introduction

The relationship between the power-handling capability of a transformer and the area product, A_p can be derived as follows.

Faraday's Law expressed in metric units is:

$$V = K_f f N_p A_c B_m \left(10^{-4}\right) \quad [21\text{-C1}]$$

Where:

K_f = 4.0 for a square wave.
K_f = 4.44 for a sine wave.

The winding area of a transformer is fully utilized when:

$$K_u W_a = N_p A_{wp} + N_s A_{ws} \quad [21\text{-C2}]$$

By definition the wire area is:

$$A_w = \frac{I}{J}, \quad [\text{cm}^2] \quad [21\text{-C3}]$$

Rearranging the equation shows:

$$K_u W_a = N_p \left(\frac{I_p}{J}\right) + N_s \left(\frac{I_s}{J}\right) \quad [21\text{-C4}]$$

Now, substitute in Faraday's Equation:

$$K_u W_a = \frac{V_p \left(10^4\right)}{A_c B_{ac} f K_f}\left(\frac{I_p}{J}\right) + \frac{V_s \left(10^4\right)}{A_c B_{ac} f K_f}\left(\frac{I_s}{J}\right) \quad [21\text{-C5}]$$

Rearranging shows:

$$W_a A_c = \frac{\left[\left(V_p I_p\right) + \left(V_s I_s\right)\right]\left(10^4\right)}{B_{ac} f J K_f K_u}, \quad [\text{cm}^4] \quad [21\text{-C6}]$$

The output power, P_o, is:

$$P_o = V_s I_s, \quad \text{[watts]} \quad [21\text{-C7}]$$

The input power, P_{in}, is:

$$P_{in} = V_p I_p, \quad \text{[watts]} \quad [21\text{-C8}]$$

Then:

$$P_t = P_{in} + P_o, \quad \text{[watts]} \quad [21\text{-C9}]$$

Therefore:

$$W_a A_c = \frac{P_t\left(10^4\right)}{B_{ac} f J K_f K_u}, \quad [\text{cm}^4] \quad [21\text{-C10}]$$

By definition:

$$A_p = W_a A_c \quad [21\text{-C11}]$$

Then:

$$A_p = \frac{P_t\left(10^4\right)}{B_{ac} f J K_f K_u}, \quad [\text{cm}^4] \quad [21\text{-C12}]$$

Inductor Derivation for the Core Geometry, K_g

Introduction

Inductors, like transformers, are designed for a given temperature rise. They can also be designed for a given regulation. The regulation and energy-handling ability of a core are related to two constants, K_g and K_e, by the equation:

$$(\text{Energy})^2 = K_g K_e \alpha, \quad [21\text{-D1}]$$

Where:

$$\alpha = \text{Regulation}, \quad [\%]$$

The constant, K_g, is a function of the core geometry:

$$K_g = f\left(A_c, W_a, \text{MLT}\right) \quad [21\text{-D2}]$$

The constant, K_e, is a function of the magnetic and electrical operating conditions:

$$K_e = g\left(P_o, B_m\right) \quad [21\text{-D3}]$$

The derivation of the specific functions for, K_g and K_e, is as follows: First, assume a dc inductor could be an input or output as schematically shown in Figure 21-D1. The inductor resistance is R_L.

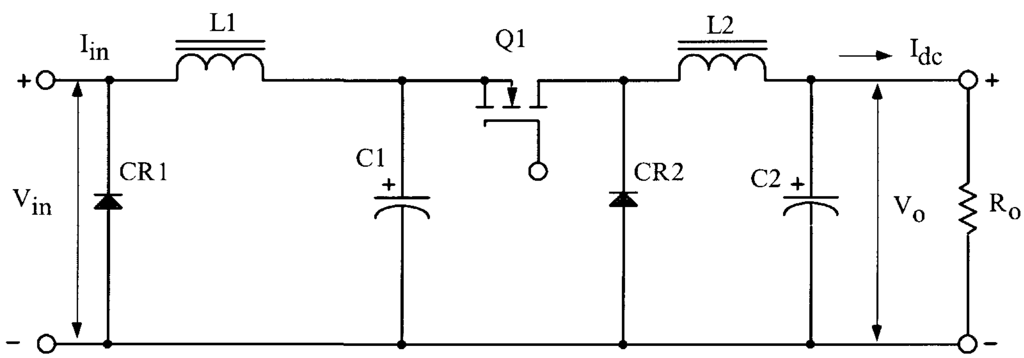

Figure 21-D1. Typical Buck Type Switching Converter.

The output power is:

$$P_o = I_{dc} V_o, \quad [\text{watts}] \quad [21\text{-D4}]$$

$$\alpha = \frac{I_{dc} R_L}{V_o}(100), \quad [\%] \quad [21\text{-D5}]$$

The inductance equation is:

$$L = \frac{0.4\pi N^2 A_c \left(10^{-8}\right)}{l_g}, \quad [\text{henrys}] \quad [21\text{-D6}]$$

The inductor flux density is:

$$B_{dc} = \frac{0.4\pi N I_{dc}\left(10^{-4}\right)}{l_g}, \quad \text{[tesla]} \quad [21\text{-D7}]$$

Combine Equations [21-D6] and [21-D7]:

$$\frac{L}{B_{dc}} = \frac{N A_c\left(10^{-4}\right)}{I_{dc}} \quad [21\text{-D8}]$$

Solve for N:

$$N = \frac{L I_{dc}\left(10^{4}\right)}{B_{dc} A_c}, \quad \text{[turns]} \quad [21\text{-D9}]$$

From the resistivity formula, it is easily shown that:

$$R_L = \frac{(\text{MLT}) N_p^2}{W_a K_u}\rho, \quad \text{[ohms]} \quad [21\text{-D10}]$$

Where:

$\rho = 1.724\ (10^{-6})$ ohm cm

Combining Equations [21-D5] and [21-D10]:

$$\alpha = \left(\frac{I_{dc}}{V_o}\right)\left(\frac{(\text{MLT}) N_p^2}{W_a K_u}\rho\right)(100), \quad [\%] \quad [21\text{-D11}]$$

Take Equation [21-D9] and square it:

$$N^2 = \left(\frac{L I_{dc}}{B_{dc} A_c}\right)^2 \left(10^{8}\right) \quad [21\text{-D12}]$$

Combine Equations [21-D11] and [21-D12]:

$$\alpha = \left(\frac{I_{dc}(\text{MLT})}{V_o W_a K_u}\rho\right)\left(\frac{L I_{dc}}{B_{dc} A_c}\right)^2 \left(10^{10}\right) \quad [21\text{-D13}]$$

Combine and simplify:

$$\alpha = \left(\frac{I_{dc}(\text{MLT})(L I_{dc})^2}{V_o W_a K_u B_{dc}^2 A_c^2}\rho\right)\left(10^{10}\right) \quad [21\text{-D14}]$$

Multiply the equation by I_{dc} / I_{dc} and combine:

$$\alpha = \left(\frac{(\text{MLT})(LI_{dc}^2)^2}{V_o I_{dc} W_a K_u B_{dc}^2 A_c^2} \rho \right) (10^{10}) \quad [21\text{-D15}]$$

The energy equation is:

$$\text{Energy} = \frac{LI_{dc}^2}{2}, \quad [\text{watt-second}] \qquad [21\text{-D16}]$$
$$2\text{Energy} = LI_{dc}^2$$

Combine and simplify:

$$\alpha = \left(\frac{(2\text{Energy})^2}{P_o B_{dc}^2} \right) \left(\frac{\rho(\text{MLT})}{W_a K_u A_c^2} \right) (10^{10}) \quad [21\text{-D17}]$$

The resistivity is:

$$\rho = 1.724\,(10^{-6}) \ [\text{ohm cm}] \quad [21\text{-D18}]$$

Combine the resistivity:

$$\alpha = \left(\frac{6.89(\text{Energy})^2}{P_o B_{dc}^2} \right) \left(\frac{(\text{MLT})}{W_a K_u A_c^2} \right) (10^{4}) \quad [21\text{-D19}]$$

Solving for energy:

$$(\text{Energy})^2 = 0.145 P_o B_{dc}^2 \left(\frac{W_a A_c^2 K_u}{\text{MLT}} \right) (10^{-4}) \alpha \quad [21\text{-D20}]$$

The core geometry equals:

$$K_g = \frac{W_a A_c^2 K_u}{\text{MLT}}, \quad [\text{cm}^5] \quad [21\text{-D21}]$$

The electrical conditions:

$$K_e = 0.145 P_o B_{dc}^2 (10^{-4}) \quad [21\text{-D22}]$$

The regulation and energy-handling ability is:

$$(\text{Energy})^2 = K_g K_e \alpha \quad [21\text{-D23}]$$

The copper loss is:

$$\alpha = \frac{P_{cu}}{P_o}(100), \quad [\%] \quad [21\text{-D24}]$$

Inductor Derivation for the Area Product, A_p

Introduction

The energy-handling capability of an inductor can be determined by the area product, A_p. The area product, A_p, relationship is obtained by the following: (Note that symbols marked with a prime (such as H`), are mks (meter-kilogram-second) units.)

$$E = L\frac{dI}{dt} = N\frac{d\phi}{dt} \quad [21\text{-E1}]$$

Combine and simplify:

$$L = N\frac{d\phi}{dI} \quad [21\text{-E2}]$$

Flux density is:

$$\phi = B_m A_c^{'} \quad [21\text{-E3}]$$

$$B_m = \frac{\mu_o N I}{l_g^{'} + \left(\frac{\text{MPL}'}{\mu_\text{m}}\right)} \quad [21\text{-E4}]$$

$$\phi = \frac{\mu_o N I A_c^{'}}{l_g^{'} + \left(\frac{\text{MPL}'}{\mu_\text{m}}\right)} \quad [21\text{-E5}]$$

$$\frac{d\phi}{dI} = \frac{\mu_o N A_c^{'}}{l_g^{'} + \left(\frac{\text{MPL}'}{\mu_\text{m}}\right)} \quad [21\text{-E6}]$$

Combine Equations [21-E2] and [21-E6]:

$$L = N\frac{d\phi}{dI} = \frac{\mu_o N^2 A_c^{'}}{l_g^{'} + \left(\frac{\text{MPL}'}{\mu_\text{m}}\right)} \quad [21\text{-E7}]$$

The energy equation is:

$$\text{Energy} = \frac{LI^2}{2}, \quad \text{[watt-seconds]} \quad [21\text{-E8}]$$

Combine Equations [21-E7] and [21-E8]:

$$\text{Energy} = \frac{LI^2}{2} = \frac{\mu_o N^2 A_c' I^2}{2\left(l_g' + \left(\frac{\text{MPL}'}{\mu_m}\right)\right)} \quad [21\text{-E9}]$$

If B_m is specified:

$$I = \frac{B_m\left(l_g' + \left(\frac{\text{MPL}'}{\mu_m}\right)\right)}{\mu_o N} \quad [21\text{-E10}]$$

Combine Equations [21-E7] and [21-E10]:

$$\text{Energy} = \frac{\mu_o N^2 A_c'}{2\left(l_g' + \left(\frac{\text{MPL}'}{\mu_m}\right)\right)}\left(\frac{B_m\left(l_g' + \left(\frac{\text{MPL}'}{\mu_m}\right)\right)}{\mu_o N}\right)^2 \quad [21\text{-E11}]$$

Combine and simplify:

$$Energy = \frac{B_m^2\left(l_g' + \left(\frac{\text{MPL}'}{\mu_m}\right)\right)A_c'}{2\mu_o} \quad [21\text{-E12}]$$

The winding area of a inductor is fully utilized when:

$$K_u W_a' = N A_w' \quad [21\text{-E13}]$$

By definition the wire area is:

$$A_w' = \frac{I}{J'} \quad [21\text{-E14}]$$

Combining Equations [21-E13] and [21-E14]:

$$K_u W_a' = N\left(\frac{I}{J'}\right) \quad [21\text{-E15}]$$

Solving for I:

$$I = \frac{K_u W_a J}{N} = \frac{B_m\left(l_g' + \left(\frac{\text{MPL}'}{\mu_m}\right)\right)}{\mu_o N} \quad [21\text{-E16}]$$

Rearrange Equation [21-E16]:

$$l_g' + \left(\frac{\text{MPL}'}{\mu_\text{m}}\right) = \frac{K_u W_a' J' \mu_o}{B_m} \quad [21\text{-E17}]$$

Now, substitute in Energy Equation [21-E11]:

$$\text{Energy} = \frac{B_m^2 \left(\frac{K_u W_a' J' \mu_o}{B_m}\right) A_c'}{2\mu_o} \quad [21\text{-E18}]$$

Rearrange Equation [21-E18]:

$$\text{Energy} = \left(\frac{B_m^2 A_c'}{2\mu_o}\right)\left(\frac{K_u W_a' J' \mu_o}{B_m}\right) \quad [21\text{-E19}]$$

Combine and simplify:

$$\text{Energy} = \left(\frac{B_m K_u W_a' J' A_c'}{2}\right) \quad [21\text{-E20}]$$

Now, multiply mks units to return cgs.

$$W_a' = W_a\left(10^{-4}\right)$$
$$A_c' = A_c\left(10^{-4}\right)$$
$$J' = J\left(10^{4}\right)$$
$$\text{MPL}' = \text{MPL}\left(10^{-2}\right)$$
$$l_g' = l_g\left(10^{-2}\right)$$

We can substitute into the energy equation to obtain:

$$\text{Energy} = \frac{B_m K_u W_a J A_c}{2}\left(10^{-4}\right) \quad [21\text{-E21}]$$

Solve for the area product:

$$A_p = W_a A_c$$
$$A_p = \frac{2(\text{Energy})}{B_m J K_u}, \quad [\text{cm}^4] \quad [21\text{-E22}]$$

Transformer Regulation

The minimum size of a transformer is usually determined either by a temperature rise limit, or by allowable voltage regulation, assuming that size and weight are to be minimized. Figure 21-F1 shows a circuit diagram of a transformer with one secondary.

Note that α = regulation (%).

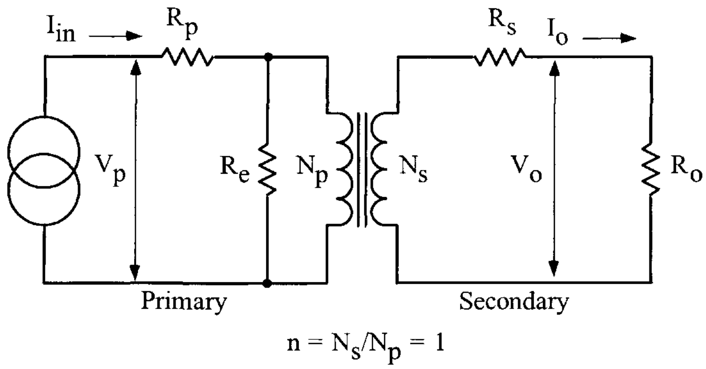

Figure 21-F1. Transformer Circuit Diagram.

The assumption is that distributed capacitance in the secondary can be neglected because the frequency and secondary voltage are not excessively high. Also, the winding geometry is designed to limit the leakage inductance to a level low enough to be neglected under most operating conditions. The transformer window allocation is shown in Figure 21-F2.

$$\frac{W_a}{2} = \text{Primary} = \text{Secondary} \quad [21\text{-F1}]$$

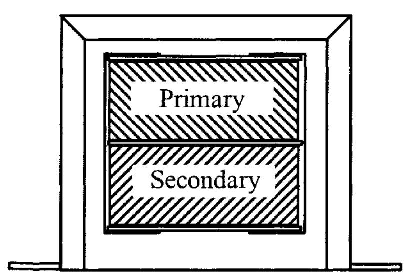

Figure 21-F2. Transformer Window Allocation.

Transformer voltage regulation can now be expressed as:

$$\alpha = \frac{V_o(\text{N.L.}) - V_o(\text{F.L.})}{V_o(\text{F.L.})}(100), \quad [\%] \quad [21\text{-F2}]$$

In which, V_O(N.L.) is the no load voltage, and V_O(F.L.) is the full load voltage. For the sake of simplicity, assume the transformer in Figure 21-F1 is an isolation transformer, with a 1:1 turns ratio, and the core impedance, R_e, is infinite.

If the transformer has a 1:1 turns ratio and the core impedance is infinite, then:

$$I_{in} = I_o, \quad \text{[amps]}$$

$$R_p = R_s, \quad \text{[ohms]} \quad \text{[21-F3]}$$

With equal window areas allocated for the primary and secondary windings, and using the same current density, J:

$$\Delta V_p = I_{in} R_p = \Delta V_s = I_o R_s, \quad \text{[volts]} \quad \text{[21-F4]}$$

Regulation is then:

$$\alpha = \frac{\Delta V_p}{V_p}(100) + \frac{\Delta V_s}{V_s}(100), \quad [\%] \quad \text{[21-F5]}$$

Multiply the equation by currents, I:

$$\alpha = \frac{\Delta V_p I_{in}}{V_p I_{in}}(100) + \frac{\Delta V_s I_o}{V_s I_o}(100), \quad [\%] \quad \text{[21-F6]}$$

Primary copper loss is:

$$P_p = \Delta V_p I_{in}, \quad \text{[watts]} \quad \text{[21-F7]}$$

Secondary copper loss is:

$$P_s = \Delta V_s I_o, \quad \text{[watts]} \quad \text{[21-F8]}$$

Total copper loss is:

$$P_{cu} = P_p + P_s, \quad \text{[watts]} \quad \text{[21-F9]}$$

Then, the regulation equation can be rewritten to:

$$\alpha = \frac{P_{cu}}{P_o}(100), \quad [\%] \quad \text{[21-F10]}$$